Balanced Crossover Operators in Genetic Algorithms

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Abstract

In several combinatorial optimization problems arising in cryptography and design theory, the admissible solutions must often satisfy a balancedness constraint, such as being represented by bitstrings with a fixed number of ones. For this reason, several works in the literature tackling these optimization problems with Genetic Algorithms (GA) introduced new balanced crossover operators which ensure that the offspring has the same balancedness characteristics of the parents. However, the use of such operators has never been thoroughly motivated, except for some generic considerations about search space reduction.

In this paper, we undertake a rigorous statistical investigation on the effect of balanced and unbalanced crossover operators against three optimization problems from the area of cryptography and coding theory: nonlinear balanced Boolean functions, binary Orthogonal Arrays (OA) and bent functions. In particular, we consider three different balanced crossover operators, two of which have never been published before, and compare their performances with classic one-point crossover. The statistical comparison shows that for the problems of nonlinear balanced Boolean functions and binary OA the use of balanced crossover operators gives GA a definite advantage over one-point crossover. For the case of bent functions, the situation is reversed, with the unbalanced crossover providing the best performances.

Keywords genetic algorithms, crossover operators, balanced bitstrings, Boolean functions, orthogonal arrays, bent functions

1 Introduction

Crossover (or *recombination*) operators play a crucial role in Genetic Algorithms (GA). The idea underlying crossover, borrowed from biological evolution, is quite simple: given two candidate solutions, combining parts of their chromosomes will yield an offspring potentially having better fitness than the parents. This strategy stands on the observation that fit individuals share some traits encoded at the chromosome level, which can be inherited by their children via crossover. Indeed, this intuition has been formalized by Holland [6] with the concept of *building blocks* used in *schema theory*.

There exist several classes of combinatorial optimization problems whose feasible solutions must contain a specified number of ones, i.e. they must have a *fixed Hamming weight*. Examples of such problems come, for instance, from the domain of *cryptography*, where *balanced Boolean functions* are used to design symmetric key cryptosystems. Another research area where balanced binary strings are sought is that of *combinatorial designs*: there, one is interested in constructing subsets of a certain support space (called *blocks*) which satisfy specific balancedness constraints. A third research field where fixed-weight bitstrings are used is that of *portfolio optimization*; indeed, a portfolio can be represented by a binary vector where the positions set to 1 indicate that the selected assets.

Genetic algorithms seem like a sensible choice for solving the optimization problems mentioned above. However, breeding feasible solutions which have a fixed Hamming weight is something that classic GA recombination operators such as *one-point* crossover cannot handle. As a matter of fact, starting from two individuals with the same number of ones and applying one-point crossover will likely produce an offspring having a different Hamming weight. This is due to the fact that one-point crossover (as well as most other recombination operators in the literature) does not enforce any control over the multiplicities of the alleles copied in the offspring. Of course, this drawback in dealing with fixed-weight bitstrings can be addressed at the fitness function level. Since we do not have any guarantee that the offspring has the desired number of ones, the idea is to add a *penalty factor* to the fitness function which punishes deviations from the expected Hamming weight. Although being the simplest solution to cope with this constraint, one might argue that it wastes a lot of fitness evaluations, because most of the solutions generated by classic crossover operators will violate the fixed-weight property.

An alternative way to address this problem is to design new recombination operators that *preserve* the Hamming weight of the bitstrings, which we term *balanced crossover operators* in what follows. The first researchers who pioneered this approach in the area of cryptography were Millan et al. [12], who proposed a counter-based crossover operator to evolve balanced Boolean functions, which was later adapted to evolve *plateaued function* in [8]. Similar operators have been later proposed for GA applied to combinatorial designs problems [9, 10], portfolio optimization [2, 3], multiobjective k-subset selection [11] and disease classification [15, 16].

Looking at the existing literature, one can remark that the introduction of balanced crossover operators has never been thoroughly motivated. In fact, the only recurring motivation supporting the use of such operators is the reduction of the search space (see e.g. [10]). To be sure, restraining the crossover operator to produce only fixed Hamming weight bitstrings greatly shrinks the space of candidate solutions searched by GA. Nonetheless, although the reduction is quite evident for short strings, this advantage becomes less clear as the length n of the bitstrings get bigger and and the Hamming weight approaches $n/2$, since in this case $\binom{n}{k}$ is asymptotically equivalent to 2^n. Moreover, most of the works in the literature employing balanced crossover operators do not perform a sound comparison of their results with those that can be obtained with classic operators. Hence, it is not even clear on a statistical basis whether balanced crossover operators actually bring any advantage to GA working with fixed Hamming weight bitstrings.

The aim of this paper is to begin closing this gap by performing a thorough statistical comparison of balanced and classic crossover operators over a set of problems from the area of cryptography and design theory. In particular, we consider three balanced crossover operators in our investigation: the first is a modification of the counter-based operator proposed by Millan et al. [12]. The other two, as far as our knowledge goes, has never been published before, and they are based respectively on the *map of ones* and *zero lengths* chromosome encodings. On the other hand, as a term for comparison we considered one-point crossover, optimizing the Hamming weight is as a penalty factor in the fitness function.

We considered three combinatorial optimization problems in our statistical investigation. The first one regards *nonlinear balanced Boolean functions*, where the goal is to maximize the nonlinearity of the functions while retaining their balancedness. The second problem, always concerning Boolean functions, is the evolution of *bent functions*, which reach the highest possible nonlinearity and, although unbalanced, have a specified Hamming weight. Finally, the third problem pertains binary *Orthogonal Arrays* (OA), which are Boolean matrices having balanced subsets of columns.

We carried out our experiments over three different instances for each problem. In order to compare the performances of the four crossover operators, we employed a *non-parametric* statistical test over the best individual produced by each experimental run, namely the *Mann-Whitney-Wilcoxon test* [4]. Some works in the literature [13, 14] performed a comparison with non-parametric test on classic crossover operators, but did not consider balanced operators. The results of our statistical analysis show that in the case of balanced nonlinear Boolean functions and binary OA balanced operators generally outperforms one-point crossover. For bent functions the situation is reversed, showing that in some cases keeping the search space unconstrained can be advantageous.

The remainder of this paper is structured as follows. Section 2 covers some basic background definitions about fixed Hamming weight bitstrings, and describe in detail the three balanced crossover operators investigated in our study. Section 3 formally states the three optimization problems considered in our investigation. Section 4 describes the experimental design of our study, discussing in particular the structure of the steady state GA employed in our experiments. Section 5 presents the results of our experiments, analyzing the performances of the four considered crossover operators through non-parametric tests. Finally, Section 6 puts the obtained results in perspective, and sketches some possible future directions of research on the subject.

2 Balanced Crossover Operators

In this section, we describe the three balanced crossover operators analyzed in our experiments. Before delving into the details of each operator, we first recall some basic definitions and results about bitstrings and their Hamming weights.

Let $\mathbb{F}_2 = \{0, 1\}$ be the finite field with two elements. A *bitstring* of length $n \in \mathbb{N}$ is a binary vector x of n components, each of them belonging to $\mathbb{F}_2$. We denote by $\mathbb{F}_2^n$ the set of all bitstrings of length n. In what follows, we will often endow $\mathbb{F}_2^n$ with a vector space structure, with bitwise XOR (denoted as $\oplus$) as vector sum and logical AND as multiplication by a scalar from $\mathbb{F}_2$. Given a bitstring $x \in \mathbb{F}_2^n$, let $supp(x) = \{i : x_i \neq 0\}$ be the *support* of x, that is, the set of coordinates set to 1 of the bitstring. The *Hamming weight* $w_H(x)$ of x is then defined as the cardinality of its support, i.e. $w_H(x) = |supp(x)|$. If n is even and $w_H(x) = n/2$, we say that the bitstring x is *balanced*. In other words, x is balanced when it is composed of an equal number of zeros and ones.

A binary string $x \in \mathbb{F}_2^n$ can be interpreted as the *characteristic function* of a set $S \subseteq [n] = \{1, \cdots, n\}$. In particular, the support of x corresponds exactly to S, i.e. to the image of its characteristic function. Basic combinatorial arguments show that the number of all bitstrings of length n is $|\mathbb{F}_2^n| = 2^n$, that is, the cardinality of the power set $\mathcal{P}([n])$ of $[n]$. Likewise, for $k \in [n]$ the size of the set $\mathcal{B}_{n,k}$ of bitstrings having Hamming weight k, or equivalently the number of k-subsets of $[n]$, is $\binom{n}{k}$, since it corresponds to the number of ways one can choose k objects out of n.

The search space of interest for our investigation is precisely $\mathcal{B}_{n,k}$, which we will also call the set of (n, k)-combinations in what follows. In particular, $\mathcal{B}_{n,k}$ will represent the set of feasible solutions to a particular optimization problem explored by GA. Although several types of crossover operators have been proposed in the literature of GA, very few of them consider restrictions on the Hamming weights of the chromosomes, i.e. which actually restrict the search of a GA to $\mathcal{B}_{n,k}$. More generally, such crossover operators are not specifically designed to evolve (n, k)-combinations.

We now describe the crossover operators adopted in our experiments. Each of these operators is based on a different encoding for the chromosome of a candidate solution, which corresponds to a specific representation of an (n, k)-combination. The reader is referred to Knuth [7] for further information about the properties of these encodings. We emphasize that, while the counter-based crossover operator is an adaptation of the one conceived by Millan et al. [12], to our knowledge the other two operators have not been proposed before in the literature.

2.1 Counter-Based Crossover

As discussed above, the *binary vector coding* is the most obvious and straightforward way to represent a (n, k)-combination: given a bitstring $x = (x_1, \cdots, x_n)$ of length n, the positions of x having value 1 denote the k selected objects out of a set of n, while the remaining $n - k$ zeros represent the unselected objects. As an example, consider the case where $n = 8$ and $k = 4$. A $(8, 4)$-combination can be represented by a balanced bitstring of length 16, such as: $x = (0, 10, 0, 0, 1, 1, 0, 1)$.

Of course, binary vector coding is also the most natural chromosome representation for GA. However, in order to evolve only individuals with a fixed Hamming weight, one has to come up with a particular crossover operator. Perhaps the simplest way to design such an operator is to randomly select bit-by-bit the allele from the first or the second parent to be copied in the offspring (as in uniform crossover), and use counters to keep track of the multiplicities of ones and zero in the child. When one of the two counters reaches the prescribed threshold (i.e. k for the ones counter and $n-k$ for the zero counter), the child is filled the complementary value.

To our knowledge, Millan et al. [12] were the first to propose a crossover operator based on this idea to evolve nonlinear balanced Boolean functions. We report in Algorithm 1 the pseudocode of a slightly modified operator, which we used in our experiments. Given two bitstrings $p_1, p_2 \in \mathbb{F}_2^n$ of length n and

Algorithm 1 COUNTER-CROSS(p_1, p_2, n, k)

```
s := 0; t := 0; c := 0^n;
for i := 1 to n do
  if (s = k) then
    c[i] := 0
  else
    if (t = n − k) then
      c[i] := 1
    else
      c[i] := RANDOM(p1[i], p2[i])
      if (c[i] = 1) then
        s := s + 1
      else
        t := t + 1
      end if
    end if
  end if
end for
return c
```

Hamming weight k, the procedure COUNTER-CROSS initializes the counters (s for the number of 1s and t for the number of 0s) and sets to zero all the bits in the string c, which will hold the chromosome of the child produced by the crossover operation. Then, for $i \in \{1, \cdots, n\}$, the i-th bit of c is determined as follows. If the maximum number of ones (respectively, zeros) allowed has already been reached, then $c[i]$ is set to 0 (respectively, 1). In all other cases, $c[i]$ is chosen by randomly selecting with uniform probability the i-th bit of p_1 or p_2, and the counters are updated according to the drawn value. In this way, the child c produced by COUNTER-CROSS is itself balanced.

2.2 Map of Ones Crossover

Suppose that $x = (x_1, \cdots, x_n)$ is the binary vector representation of a (n, k)-combination, and denote by $supp(x)$ its support. The *map of ones* of x is the k-dimensional vector $q = (q_1, \cdots, q_k)$ where for all $i \in \{1, \cdots, k\}$ it results that $q_i \in supp(x)$. In other words the map of ones of x corresponds to its support in vector form.

Thus, the map of ones representation lists the nonzero coordinates in the binary coding of a (n, k)-combination. Following the example of the previous section, the map of ones corresponding to the binary string x representing a $(8, 4)$-combination is $q = (2, 5, 6, 8)$, where the positions of the ones are listed in increasing order. Strictly speaking, the order of the positions is irrelevant, since they always yield the same binary representation. In the rest of the paper, however, for ease of notation we assume that all map of ones vectors are sorted in increasing order.

One can notice that the only constraint in the map of ones is that there cannot be duplicate positions in the vector. Thus, given two bitstrings of length n and weight k represented by their maps of ones, the crossover operator must be aware of the common positions between them, in order to avoid duplications. Algorithm 2 reports the pseudocode for our crossover operator. Let us suppose

Algorithm 2 MAP-1-CROSS(p_1, p_2, k)

```
c := 0^k
comm_list = FIND-COMMON-POS(p1, p2)
for i := 1 to k do
   cpar := RANDOM(p1, p2)
   cpos := RAND-POS(cpar)
   c[i] := cpar[cpos]
   REMOVE(cpar, cpar[cpos])
   if (CONTAINS(comm_list, cpar[cpos])) then
      if (cpar = p1) then
         REMOVE(p2, cpar[cpos])
      else
         REMOVE(p1, cpar[cpos])
      end if
   end if
end for
return c
```

that we have two bitstrings $x_1, x_2 \in \mathcal{B}_{n,k}$ of length n and Hamming weight k, represented respectively by the maps of ones p_1 and p_2 of length k. The procedure MAP-1-CROSS begins by initializing the map of ones of the child c and by finding the positions which p_1 and p_2 have in common. The latter operation is performed by the subroutine FIND-COM-POS, which returns the vector *comm_list*. Successively, for all $i \in \{1, \cdots, k\}$, the value $c[i]$ is computed as follows. One of the two parents is randomly chosen by calling the procedure RANDOM on p_1 and p_2. Then, a random index *cpos* is selected from the candidate parent *cpar*, and the value of $c[i]$ is set equal to *cpar*[*cpos*]. In other words, the child c inherits from the parent *cpar* the position of the 1 specified by the value *cpar*[*cpos*]. Finally, in order to avoid that the same position is selected in the next iterations, the value *cpar*[*cpos*] is removed from the candidate parent by using the REMOVE procedure. The value is also removed from the unselected parent if it is contained in *comm_list*.

2.3 Zero Lengths Crossover

Given the bitstring $x = (x_1, \cdots, x_n)$ of a (n, k)-combination, the *zero lengths coding* of x is the vector $r = (r_1, \cdots, r_{n-k+1})$ which lists the *distances between consecutive ones* in x. In other words, the values r_i denote the lengths of the runs of zeros which separate the ones in the binary vector coding, with the particular cases of r_1 and r_{n-k+1} which represent the number of zeros preceding the first 1 and following the last 1 in x, respectively.

Clearly, in order to ensure that a given zero lengths coding vector $r = (r_1, \cdots, r_{n-k+1})$ represents a valid (n,k)-combination, the following relation must hold:

$$\sum_{i=1}^{n-k+1} r_i = n - k \ . \tag{1}$$

Following the example adopted in the previous two sections, the run length coding of the bitstring x is $r = (1, 2, 0, 1, 0)$. As we pointed out in Equation (1), the zero lengths vector of a bitstring of length n and Hamming weight k is valid if and only if the sum of the components in the vector equals k. In a crossover operator based on the zero lengths representation it is thus necessary to control the sum of the run lengths of zeros in the offspring, while the components of the vector are copied from the parents. The pseudocode for the crossover operator that we designed for this specific coding is reported in Algorithm 3. The operator

Algorithm 3 ZERO-LENGTHS-CROSS(p_1, p_2, k)

```
sumz := 0
c := 0^{k+1}
for i := 1 to k do
  if (sumz = k) then
    c[i] := 0
  else
    cpar := RANDOM(p1, p2)
    c[i] := cpar[i]
    sumz := sumz + cpar[i]
  end if
end for
c[k + 1] := k − sumz
return c
```

takes as input the zero lengths vectors p_1, p_2 of two bitstrings $x_1, x_2 \in \mathcal{B}_{n,k}$, and their Hamming weight k. The first steps are devoted to the initialization of the zero length vector of the child c (filled with $k+1$ zeros) and the accumulator *sumz* used to control the value of the sum of zeros in c. The FOR loop cycles over the first k positions of c. For each iteration i, an IF block initially checks whether the sum of zeros in c has already reached k, in which case the value of $c[i]$ is set to zero. In the other case, a candidate parent *cpar* is randomly selected between p_1 and p_2 with uniform probability, and $cpar[i]$ is copied in $c[i]$. The accumulator *sumz* is also updated by adding to it the value of $cpar[i]$. After the FOR loop, the value of the last component in c is determined by simply subtracting from k the sum of zeros obtained up to that point. Thus, if *sumz* reached k in the FOR loop, the last component will be set to zero, otherwise it will contain the number of zeros necessary to "pad" the bitstring encoded by c after its last 1.

3 Optimization Problems

We now give the formal statement of the three combinatorial optimization problems that we addressed in our statistical comparison of balanced crossover operators.

3.1 Nonlinear Balanced Boolean Functions

A *Boolean function* of $n \in \mathbb{N}$ variables is a map $f : \mathbb{F}_2^n \to \mathbb{F}_2$. The common way for representing a Boolean function f is by means of its *truth table* Ω_f, which is basically a binary vector of length 2^n that specifies for each input vector $x \in \mathbb{F}_2^n$ the output value of $f(x)$, in lexicographic order. A Boolean function is called *balanced* if its *truth table* Ω_f is composed of an equal number of ones and zeros, i.e. it represents a $(2^n, 2^{n-1})$-combination.

Another representation of Boolean functions $f : \mathbb{F}_2^n \to \mathbb{F}_2$ used in cryptography is the *Walsh transform*, which is the function $W_f : \mathbb{F}_2^n \to \mathbb{Z}$ defined for all $\omega \in \mathbb{F}_2^n$ as:

$$W_f(\omega) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)} \cdot (-1)^{\omega \cdot x} \ , \tag{2}$$

where $\omega \cdot x = \omega_1 x_1 \oplus \omega_2 x_2 \oplus \cdots \oplus \omega_n x_n$ is the *scalar product* modulo 2 between the vectors $\omega, x \in \mathbb{F}_2^n$. The *spectral radius* $W_{max}(f)$ of a Boolean function f is defined as the maximum absolute value of its Walsh transform, i.e. $W_{max}(f) = \max_{\omega \in \mathbb{F}_2^n}\{|W_f(\omega)\}$.

The *nonlinearity* of a Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is defined as the minimum Hamming distance of its truth table Ω_f from the set of truth tables of all linear functions, i.e. those functions whose algebraic expressions contain only XOR. This can be computed through the following formula based on the Walsh transform:

$$Nl(f) = 2^{n-1} - \frac{1}{2} \cdot W_{max}(f) \ . \tag{3}$$

In cryptography, Boolean functions which are both balanced and have high nonlinearity play a fundamental role in the design of stream and block ciphers [1]. Since the set of all Boolean functions is composed 2^{2^n} elements, which is not exhaustively searchable for $n > 5$, evolutionary algorithms such as GA represent a possible method for finding highly nonlinear balanced Boolean functions in a reasonable amount of time. We formally state the combinatorial optimization problem as follows:

Problem 1. *Let $n \in \mathbb{N}$. Find a Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ of n variables such that f is balanced and has maximum nonlinearity.*

In particular, given the truth table bitstring $\Omega_f \in \mathbb{F}_2^n$ of a Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ of n variables, in our experiments the fitness of f is computed with the following function:

$$\text{fit}_1(f) = Nl(f) - UNB(f) \ , \tag{4}$$

where $UNB(f) = |2^{n-1} - w_H(\Omega_f)|$ is the *unbalancedness penalty factor* which punishes the deviation of f from being a balanced function. The objective of our GA, in particular, is to *maximize* $\text{fit}_1(f)$. Of course, when using balanced crossover operators the $UNB(f)$ term is not necessary, since the candidate solutions generated by GA are already balanced functions.

3.2 Bent Functions

From Equation (3), one can see that the lower the spectral radius is, the higher the nonlinearity of a Boolean function will be. Due to *Parseval's relation* [1], the minimum spectral radius is achieved when the Walsh spectrum is uniformly divided among all 2^n vectors. This means that the Walsh coefficients must all have the same absolute value $2^{\frac{n}{2}}$, thus giving the following upper bound on nonlinearity:

$$Nl(f) \leq 2^{n-1} - 2^{\frac{n}{2}-1} \ . \tag{5}$$

Clearly, equality in (5) can occur only if n is even, since the Walsh coefficients of a Boolean function must be integer numbers. The Boolean functions achieving this bound are called *bent*, and they have several applications in cryptography and coding theory [1].

A nice feature of the Walsh transform is that the Walsh coefficient $W_f(0)$ (where 0 denotes the null vector) is related to the Hamming weight of the truth table Ω_f as follows:

$$w_H(\Omega_f) = 2^{n-1} - \frac{1}{2} \cdot W_f(0) \ . \tag{6}$$

Since all Walsh coefficients of a bent function must be equal to $\pm 2^{\frac{n}{2}}$, this means that the Hamming weight of bent functions is either $2^{n-1} - 2^{\frac{n}{2}-1}$ or $2^{n-1} + 2^{\frac{n}{2}-1}$. Without loss of generality, one can narrow the attention only to the weight $2^{n-1} - 2^{\frac{n}{2}-1}$, since the others are obtained by simply complementing the corresponding truth tables. Hence, one can cast the search of bent function as an optimization problem over the set of bitstrings of length 2^n and weight $k = 2^{n-1} - 2^{\frac{n}{2}-1}$, which makes it amenable to GA with balanced crossover operators. For this reason, we adopted it as our second optimization problem for our investigation:

Problem 2. *Let $n \in \mathbb{N}$ be an even number. Find a Boolean function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ of n variables such that $W_f(\omega) = \pm 2^{\frac{n}{2}}$ for all $\omega \in \mathbb{F}_2^n$.*

Since each Walsh coefficient of a bent function must be equal to $\pm 2^{\frac{n}{2}}$, we defined a fitness function for Problem 2 which penalizes the deviation from this value for each coefficient, taken in absolute value. Formally, given $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$, the fitness function for f is defined as follows:

$$\mathrm{fit}_2(f) = \sqrt{\sum_{\omega \in \mathbb{F}_2^n} \left(2^{\frac{n}{2}} - |W_f(\omega)|\right)^2} + UNB(f) \ , \tag{7}$$

where the unbalancedness penalty factor this time is defined as $UNB(f) = |2^{n-1} - 2^{\frac{n}{2}-1} - w_H(\Omega_f)|$. In particular, the optimization objective is to *minimize* Equation (7), since having $\mathrm{fit}_2(f) = 0$ corresponds to the case where each Walsh coefficient equals $2^{\frac{n}{2}}$ in absolute value. As in the case of fit_1, when using balanced crossover operators the term $UNB(f)$ is not necessary.

3.3 Binary Orthogonal Arrays

Orthogonal Arrays (OA) are rectangular matrices whose submatrices satisfy a specific balancedness constraint on their rows. OA find several applications in statistics, combinatorial designs theory and cryptography [5]. In what follows, we will focus on *binary* OA, meaning that the matrices are Boolean. We formally define a binary OA as follows:

Definition 1. *Let $N, k, t, \lambda \in \mathbb{N}$ with $0 \le t \le k$. A $N \times k$ binary matrix A is called is called a binary* orthogonal array *(OA) with k columns, strength t and index λ (for short, an $OA(N, k, t, \lambda)$) if in each submatrix of N rows and t columns each binary t-uple occurs exactly λ times.*

Notice that the parameter λ of an OA is related to its strength t and number of rows N by the relation $\lambda = \frac{N}{2^t}$.

For our third optimization problem, formally defined below, we are interested in binary OA whose columns are truth tables of Boolean functions:

Problem 3. *Let $n, k, t \in \mathbb{N}$. Find k Boolean functions $f_1, \cdots, f_k : \mathbb{F}_2^n \to \mathbb{F}_2$ of n variables such that the matrix*

$$A = [\Omega(f_1)^\top, \Omega(f_2)^\top, \cdots, \Omega(f_k)^\top] \tag{8}$$

is an $OA(2^n, k, t, \lambda)$, with $\lambda = 2^{n-t}$.

Hence, the goal of Problem 3 is to find k n-variables Boolean functions such that the bitstrings of their truth tables are the columns of a binary OA with $N = 2^n$ rows and strength t.

A useful property for this problem is that any binary OA of strength t is also an OA of strength $t' < t$, for all $t' \in \{1, \cdots, t-1\}$. Taking $t' = 1$, this implies that each column of a binary OA must be a balanced bitstring of length N. Consequently, one can use a GA with balanced crossover operators to evolve candidate binary OA as a set of k balanced bitstrings. New solutions are bred by applying balanced crossover and mutation independently on the single bitstrings, thus maintaining the balancedness constraint on the single columns of the array. This is the optimization approach that was adopted by Mariot et al. [10], from which we took adopted the fitness function for our experiments. In particular, the fitness function stands on the idea of counting the *repeated t-uples* in the submatrices of an array.

Given a $N \times k$ binary matrix A, let I be a subset of t indices, and let A_I denote the $N \times t$ submatrix obtained by considering only the columns of A specified by the indices of I. For all binary t-uples $x \in \mathbb{F}_2^t$, let $A_I[x]$ denote the number of occurrences of x in A_I, and let $\delta(A_I, x)$ be the λ-*deviation* of x defined as $\delta(A_I, x) = |\lambda - A_I[x]|$. Then, the *Euclidean deviation* of A_I is defined as:

$$\Delta(A_I)_2 = \sqrt{\left(\sum_{x \in \mathbb{F}_2^t} |\lambda - A_I[x]|^2\right)} \ . \tag{9}$$

The fitness function for Problem 3 is then defined for all $2^n \times k$ binary matrix A formed by k n-variables Boolean functions as follows:

$$\text{fit}_3(A) = \sum_{I \subseteq [k]: |I| = t} \Delta(A_I)_2 + UNB(A) \ , \tag{10}$$

where the unbalancedness penalty factor $UNB(A)$ is defined as the sum of the unbalancedness of all Boolean functions $f_1, \cdots, f_k$, that is, $UNB(A) =$

$\sum_{i=1}^{k} |2^{n-1} - w_H(\Omega_{f_i})|$. As for the other two optimization problems, when using GA with balanced crossover operators this penalty factor can be dropped from the fitness function. The optimization objective is to minimize fit_3, since any binary matrix such that $\text{fit}_3(A) = 0$ corresponds to a binary $OA(2^n, k, t, \lambda)$.

4 Experimental Setting

In this section, we describe the details of the genetic algorithm used to test the three crossover operators presented in Section 2, and the parameters used to set up the experiments over the three optimization problem defined in Section 3.

4.1 Genetic Algorithm Details

The genetic algorithms adopted in this work is a *steady state* GA where a single pair of parents is drawn from the current population at each iteration. For selection, we employed a *deterministic tournament* operator where the best two out of t randomly sampled individuals are selected for crossover.

The four crossover operators considered in our investigation are classic one-point crossover and the three balanced crossover described in Section 2, namely counter-based, map of ones and zero-lengths crossover. Our GA generates a single child for each selected pair of parents, independently of the underlying crossover operator. In particular, since one-point crossover generates two children by design, our GA randomly selects only one of them with uniform probability, which is then subjected to mutation.

The mutation operator depends on the type of crossover: when one-point crossover is used, a classic bit-flip mutation operator is applied on the generated child. On the other hand, with balanced crossover operators a simple *swap-based* mutation operator is used, which swaps a pair of bits in a bitstring with mutation probability p_m. In this way, the Hamming weight of the child produced through balanced crossover is preserved. Once the child has been mutated, the GA evaluates the relevant fitness function on it. In particular, depending on the optimization problem considered, if one-point crossover is used then the full form of the fitness functions fit_1, fit_2 or fit_3 respectively described in Equations (4), (7) and (10) is used. On the opposite, when one of the three balanced crossover operators is employed, the unbalancedness penalty factor $UNB(\cdot)$ is dropped from the computation of the fitness functions. Similarly to the choice of the mutation operator, the creation of the initial population depends on the adopted crossover operator. For one-point crossover, the population is initialized at random, without controlling the Hamming weights of the generated bitstrings. Contrarily, for balanced crossover operators each chromosome is created by sequentially generating at random each bit in the bitstring, and using a counter to keep track of the number of ones. When the prescribed Hamming weight has been reached, the chromosome is filled with zeros in the remaining positions.

Our GA uses a worse-replacement elitist strategy: in particular, if the child has a better fitness value than any of its two parents, then the worse individual in the population is replaced by it. Further, since we are interested more in comparing the performances of the crossover operators than in generating optimal

Table 1: Problem instances and relative search space sizes

Problem	Instance	UNB Size	BAL Size
	$n = 6$	$2^{2^6} \approx 1.8 \cdot 10^{19}$	$\binom{64}{32} \approx 1.8 \cdot 10^{18}$
Bal-NL	$n = 7$	$2^{2^7} \approx 3.4 \cdot 10^{38}$	$\binom{128}{64} \approx 2.4 \cdot 10^{37}$
	$n = 8$	$2^{2^8} \approx 1.1 \cdot 10^{77}$	$\binom{256}{128} \approx 5.7 \cdot 10^{75}$
	$n = 6$	$2^{2^6} \approx 1.8 \cdot 10^{19}$	$\binom{64}{28} \approx 1.1 \cdot 10^{18}$
Bent	$n = 8$	$2^{2^8} \approx 1.1 \cdot 10^{77}$	$\binom{256}{120} \approx 3.5 \cdot 10^{75}$
	$n = 10$	$2^{2^{10}} \approx 1.8 \cdot 10^{308}$	$\binom{1024}{496} \approx 2.7 \cdot 10^{306}$
	$OA(16, 8, 3, 2)$	$\binom{2^{16}}{8} \approx 8.4 \cdot 10^{33}$	$\binom{\binom{16}{8}}{8} \approx 1.8 \cdot 10^{28}$
Bin-OA	$OA(16, 8, 2, 4)$	$\binom{2^{16}}{8} \approx 8.4 \cdot 10^{33}$	$\binom{\binom{16}{8}}{8} \approx 1.8 \cdot 10^{28}$
	$OA(16, 15, 2, 4)$	$\binom{2^{16}}{15} \approx 1.3 \cdot 10^{60}$	$\binom{\binom{16}{8}}{15} \approx 3.3 \cdot 10^{49}$

solutions for the considered problems, the GA terminates after it performs a specified number of fitness evaluations.

4.2 Experimental Parameters

Table 1 reports the problem instances tested in our experiments for each of the three considered optimization problems. The three problems in the table are respectively identified in the first column by the names Bal-NL for highly nonlinear balanced Boolean functions, Bent for bent functions and Bin-OA for binary orthogonal arrays. The second column reports the problem instances considered for each problem, which are characterized by the number of variables n for the problems Bal-NL and Bent and by the set of parameters $OA(2^n, k, t, \lambda)$ for the Bin-OA problem. The third and fourth columns report, for each problem instance, the size of the corresponding search space respectively when one-point crossover is used (UNB Size) and when balanced crossover operators are adopted (BAL size). The details for the computation of the search space sizes can be found in [10]. It can be remarked from Table 1 that for the two Boolean functions problems (Bal-NL and Bent) the use of balanced crossover operators yields a reduction of the search space between one ad two order of magnitude. On the other hand, for the Bin-OA problem the reduction is much more consistent.

For each problem instance, we ran our steady state GA with each of the four crossover operators for $R = 50$ experimental runs. Hence, we performed a total of $3 \cdot 4 \cdot 50 = 600$ experiments for each of the three optimization problems. Each GA experiment used a population size of $P = 50$ individuals, tournament size $t = 3$ and mutation probability $p_m = 0.2$, and stopped after $fit = 500000$ fitness evaluations.

To compare the results we employed the *Mann-Whitney-Wilcoxon test*. The alternative hypothesis used was that the two distributions were not equal. More precisely, that the probability of a sample a from the first distribution exceeding a sample b from the second distribution is different from the probability of b exceeding a. The significance value α for the statistical tests was set to 0.01.

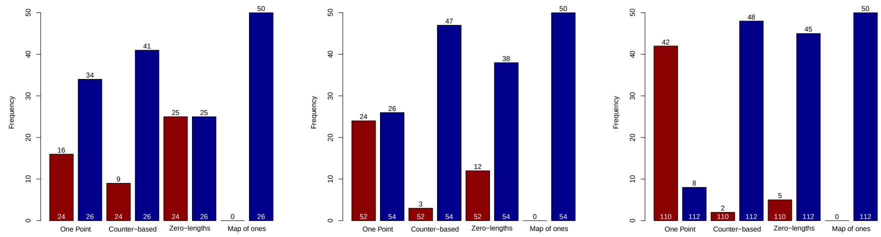

Figure 1: The results for the balanced Boolean functions problem for 6 (left), 7 (center), and 8 (right) variables.

5 Results

The results of the experiments are summarized in Figure 1 for Balanced Boolean functions, where the histograms show the distribution of the different fitness values obtained. Figure 3 summarizes the results for bent functions, and finally Figure 2 show the results for the orthogonal arrays problem. The boxplots show the median, minimum, maximum, first and third quantiles (excluding outliers) in addition to the exact distribution of the obtained results.

From the results for 6 variables, it is possible to observe that the map of ones crossover seems to produce the best results, with all fitness values obtained being equal to 26 (recall that this is a maximization problem, so higher values are better). This is evident also in the statistical tests, with a significant difference between the distribution of the "map of ones" results and all the other methods, with p-values of $1.4 \cdot 10^{-5}$, 0.0018, and $9.5 \cdot 10^{-9}$ when compared to the one point, counter-based, and zero-lengths crossover, respectively. Similar results hold for 7 variables, where the map of ones and the counter-based operators perform better than one-point crossover (p-values of $2.6 \cdot 10^{-6}$ and $2.3 \cdot 10^{-8}$, respectively). The map of ones crossover also performs better than the zero-lengths crossover (p-value of 0.0002), but no other comparison of the results gives a statistically significant difference. The results are different from the case of 8 variables, with one-point crossover, the only one not preserving balancedness, resulting in a statistically significant difference with all other operators (in all cases the p-values are less than 10^{-12}). Therefore, it appears as if the map of ones crossover is, on this problem, the best performer, but its advantage when the problem size increases is not preserved with respect to the others balanced crossovers. When the problem size increases, the inability for one point crossover to preserve balancedness is a serious drawback, making it the worst performer for 8 variables.

In the case of bent functions, it is possible to observe that for $n = 6$ variables the results show no clear "winner" (and the statistical tests confirms it, with no difference being statistically significant). There is a different behavior for larger problem sizes: for $n = 8$ and $n = 10$ variables the best performer is one-point crossover, the only unbalanced operator with a statistically significant difference (p-values of less than 10^{-11} for $n = 8$ and less than 10^{-7} for $n = 10$). The second best performer is zero-lengths crossover, with a statistically significant difference when compared to both the counter-based and the map of ones crossover (p-

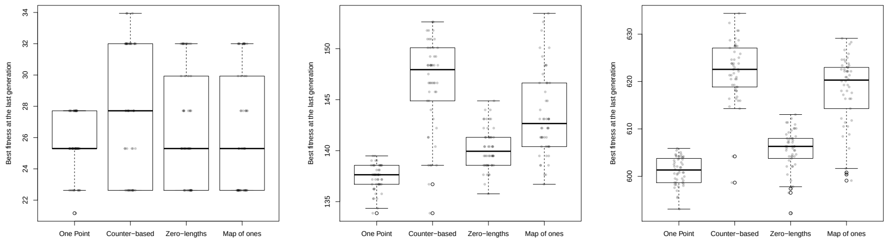

Figure 2: The results of the bent functions problem, for 6 (left), 8 (center), and 10 (right) variables.

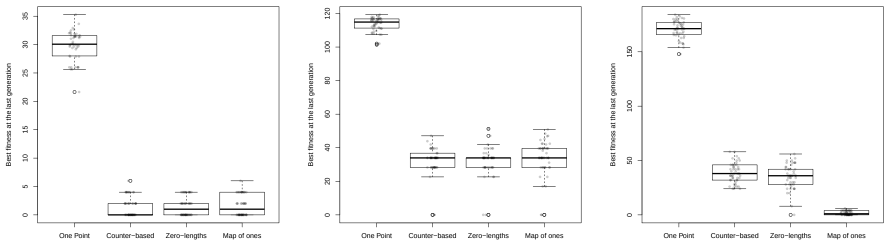

Figure 3: The results of the orthogonal arrays problem, for $N = 16, k = 8, t = 2, \lambda = 4$ (left), $N = 16, k = 8, t = 3, \lambda = 2$ (center), and $N = 16, k = 15, t = 2, \lambda = 4$ (right).

values less than 10^{-4} for $n = 8$ and $n = 10$). Finally, the "map of ones" crossover performs significantly better than the counter-based crossover, even if the p-values are higher (0.0005 for $n = 8$ and 0.0099 for $n = 10$). In conclusion, for bent functions there is a clear ranking of crossover methods, and preserving the number of ones in the function is actually a disadvantage.

In the last problem, the construction of orthogonal arrays, it is possible to observe that for all parameters the worst performer with highest fitness values (recall that this is a minimization problem) is one-point crossover. In all cases, the difference is statistically significant (with a p-value of less than 10^{-17}). The only other statistically significant difference happens for $N = 16, k = 15, t = 2, \lambda = 4$, where the map of ones crossover is the best performer, with a statistically significant difference (p-values of less then 10^{-16} when compared to both the counter-based and zero-lengths crossovers). In conclusion, in this problem the use of crossover operators preserving balancedness is to be preferred and, among them, the one performing the best is the map of ones crossover.

6 Conclusions

In this paper, we investigated the effect of three balanced crossover operators in constraining the search space explored by a steady state GA over three combinatorial optimization problems from the domains of cryptography and combinatorial designs. In particular, the counter-based crossover operator is a slightly modified version of the crossover designed by Millan et al. [12], while to the best of our knowledge the zero-lengths and map of ones crossovers are proposed for the first time in the present work.

We set up a statistical investigation, based on the Mann-Whitney-Wilcoxon test, in order to compare the performances of these three operators with that of the classic one-point crossover, which does not enforce any constraint on the Hamming weight of the bitstrings. The obtained results showed that for the problems of nonlinear balanced Boolean functions and binary orthogonal arrays the use of balanced operators gives a definite advantage over one-point crossover, with map of ones crossover being the best among the considered three. Therefore, for these two optimization problems constraining the search space explored by GA is certainly beneficial.

On the other hand, bent functions represent a remarkable exception, since in this problem one-point crossover fared considerably better than any of the three balanced operators. What could be the reason of this difference? At a first glance, the most noticeable difference between bent functions and (for instance) balanced nonlinear functions is that in the former we are considering *unbalanced* bitstrings of fixed Hamming weight, while in the latter we only deal with balanced bitstring. Hence, one could think that in the case of bent function the use of balanced operators constrains too much the search space explored by GA. However, by looking at Table 1 one can see that the differences between the sets of balanced Boolean functions of n variables and functions of weight $2^{n-1} - 2^{\frac{n}{2}-1}$ is not really huge (compare for $n = 8$, for example). What we suspect is that in the case of bent functions the constrained search space is highly irregular, making it really hard for GA to escape from local optima. We plan to investigate this issue in future research, in particular by performing an analysis of the fitness landscapes for functions of weight $2^{n-1} - 2^{\frac{n}{2}-1}$.

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