Artificial Intelligence and Security Lab
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Counting coprime polynomials... with complications

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Coprime Polynomials

Object: pairs of binary polynomials of degree $n \in \mathbb{N}$:

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \ ,$$
$$g(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + x^n \ ,$$

where $a_i, b_i \in GF(2) = \mathbb{F}_2 = \{0, 1\}$

$$f, g \in \mathbb{F}_2[x] \text{ are } \textbf{coprime} \Leftrightarrow \gcd(f, g) = 1$$

Applications of coprime pairs in cryptography and coding:

- *Discrete logarithms* in finite fields [C84]
- Decoding *alternant codes* [F95]

Euclid's Algorithm

Check if $\gcd(f, g) = 1 \Rightarrow$ **Euclid's algorithm**

Example: $n = 3, \quad f(x) = x^3 + x^2 + x + 1, \quad g(x) = x^3 + 1$

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$$f(x) = q(x) \cdot g(x) + r(x)$$

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$x^3 + x^2 + x + 1 = 1 \cdot (x^3 + 1) + (x^2 + x)$

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Compact notation:

$$(x^3 + x^2 + x + 1, x^3 + 1) \xrightarrow{1} (x^3 + 1, x^2 + x) \xrightarrow{x+1} (x^2 + x, x + 1) \xrightarrow{x} (x + 1, 0)$$

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$gcd(f,g) = x + 1 \Rightarrow (f,g)$ **not** coprime

DilcuE's Algorithm

- **Remark:** (f, g) can be recovered from $(x+1, 0)$ with the same quotients in reverse order
- Called **DilcuE's algorithm** by Benjamin and Bennett [BB07]

$(x+1, 0) \xrightarrow{x}$

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$$(x+1, 0) \xrightarrow{x} (x^2+x, x+1) \xrightarrow{x+1}$$

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- Suppose we change the **last** remainder from 0 to 1:

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- By construction, (f', g') **are coprime**

Counting by Bijection

In essence: bijection for coprime/non-coprime pairs over $\mathbb{F}_2$:

1. Apply Euclid to (f, g)
2. If the last remainder is 0, change it to 1. Otherwise, set it to the second-last remainder
3. Apply DilcuE's algorithm to the reversed quotients

Theorem ([BB07, CSWZ98, R00])

Let $f, g \in \mathbb{F}_2[x]$ of degree n be randomly chosen. Then, the probability that $\gcd(f, g) = 1$ is $\frac{1}{2}$.

In other words: the number of coprime pairs is 2^{2n-1}

Enter the complication

We require now that both f and g have a **nonzero constant term**:

$$f(x) = \mathbf{1} + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \ ,$$
$$g(x) = \mathbf{1} + b_1 x + \cdots + b_{n-1} x^{n-1} + x^n \ .$$

Problems:

1. *Count* all such pairs
2. *Enumeration algorithm*

Remark: the trick above does not work! Changing the last remainder gives no control over the final constant terms

non-coprime $\leftrightarrow$ **coprime**

$$(x^3 + x^2 + x + \mathbf{1}, x^3 + \mathbf{1}) \leftrightarrow (x^3 + x + \mathbf{1}, x^3 + x^2 \ (+\mathbf{0}))$$

... Why do we want to do that?

Orthogonal Latin Squares by Linear Cellular Automata

- Bipermutive Linear rule: $f(x) = x_1 \oplus a_1 x_2 \oplus \cdots \oplus a_{n-1} x_{n-1} \oplus x_n$
- Associated Polynomial: $P_f(X) = 1 + a_1 X + \cdots + a_{n-1} X^{n-1} + X^n$

Theorem ([MGFL20])

Two bipermutive linear CA generates orthogonal Latin squares if and only if their associated polynomials are coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1 1	4 2	3 3	2 4
2 2	3 1	4 4	1 3
4 3	1 4	2 1	3 2
3 4	2 3	1 2	4 1

(c) Superposition

Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Asking for clues (precisely 6 years ago...)

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Dear Arthur, what do you think of this complication?

Luca

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Dear Arthur, what do you think of this complication?

Luca

... One hour later...

Arthur Benjamin

Dear Luca, off the top of my head, there are $q^2 - 1$ equivalence classes, all of which are co-prime except one? But I may be wrong.

Asking for clues (precisely 6 years ago...)

Dear Arthur, what do you think of this complication?

Luca

... One hour later...

Arthur Benjamin

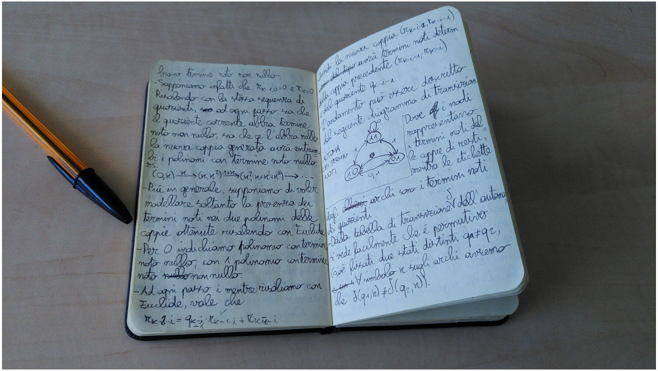
Dear Luca, off the top of my head, there are $q^2 - 1$ equivalence classes, all of which are co-prime except one? But I may be wrong.

?

One MONTH later...

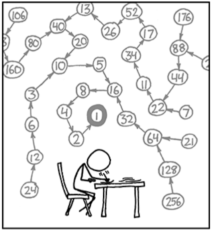
One MONTH later...

... He was indeed right! But took me several weeks to prove it



Sadly, the clue was not enough to solve the counting problem

Counting by Recurrence



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

S https://xkcd.com/710/

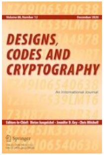
- Number of coprime polynomial pairs of degree n and nonzero constant term:

$$a(n) = 4^{n-1} + a(n-1) = \frac{4^{n-1}-1}{3}$$
$$= 0, 1, 5, 21, 85, \ldots$$

- Corresponds to OEIS A002450

- Generalized for any finite field $\mathbb{F}_q$ in [MGFL20] (but enumeration not addressed)

L. Mariot, M. Gadouleau, E. Formenti, and A. Leporati. Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)



Problem Structure

Strategy: characterize the *sequences* of quotients that gives only $(1,1)$ coprime pairs when starting from the remainders $(1,0)$

Three parts of the problem:

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Three parts of the problem:

$$
\begin{array}{lllll}
q_1 \rightarrow & \overbrace{x^{d_1}}^{\textit{degrees}} & + \overbrace{q_{1,d_1-1}x^{d_1-1} + \cdots + q_{1,1}x}^{\text{middle terms}} & + & \overbrace{s_1}^{\text{constant terms}} \\
q_2 \rightarrow & x^{d_2} & + q_{2,d_2-1}x^{d_2-1} + \cdots + q_{2,1}x & + & s_2 \\
\vdots \rightarrow & \vdots & + \quad \vdots \quad + \cdots + \quad \vdots & + & \vdots \\
q_k \rightarrow & x^{d_k} & + q_{k,d_k-1}x^{d_k-1} + \cdots + q_{k,1}x & + & s_k
\end{array}
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\vdots \rightarrow & \vdots & + \vdots + \cdots + \vdots + \vdots \\
q_k \rightarrow & x^{d_k} & + q_{k,d_k-1}x^{d_k-1} + \cdots + q_{k,1}x + s_k
\end{array}$$

Notation: $r_i, r_{i+1} \rightarrow$ consecutive remainders produced by Euclid's algorithm at step i. Step $i+1$:

$$r_i(x) = q_{i+1}(x)r_{i+1}(x) + r_{i+2}(x)$$

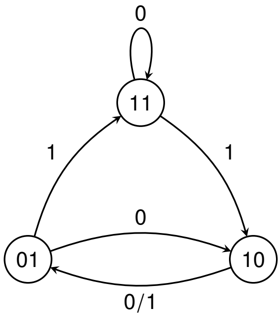
Finite State Automaton of Remainders

$$r_i(x) = q_{i+1}(x) r_{i+1}(x) + r_{i+2}(x)$$

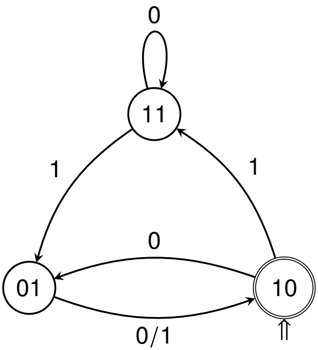
- $(c_i, c_{i+1}) \rightarrow$ constant terms of r_i and r_{i+1}
- $s_{i+1} \rightarrow$ constant term of q_{i+1}
- $\delta((c_i, c_{i+1}), s_{i+1}) \rightarrow$ *next* pair (c_{i+1}, c_{i+2})

(c_i, c_{i+1})	s_{i+1}	$\delta((c_i, c_{i+1}), s_{i+1})$
$(1,1)$	0	$(1,1)$
$(1,1)$	1	$(1,0)$
$(1,0)$	0	$(0,1)$
$(1,0)$	1	$(0,1)$
$(0,1)$	0	$(1,0)$
$(0,1)$	1	$(1,1)$

Remark: the pair $(0,0)$ *never* occurs



The Regular Language of Constant Terms Sequences



Inverse FSA

- The FSA is *permutative*: for DilcuE's, simply reverse the arrows
- **Initial state**: 10
- **Final state**: 11 (but we can use 10)

Regular Expression of the Language:

$$L = (0(0+1) + (10^*1(0+1)))^*$$

Enumeration/counting of Constant Terms Sequences

- **Enumeration**: generate all words of length k [M97]
- **Counting**: exploit *algebraic language theory*

Transform $L = (0(0+1)+(10^*1(0+1)))^*$ in a FPS as follows:

- $0,1 \Rightarrow X$
- $+,\cdot \Rightarrow +,\cdot$
- $^* \Rightarrow \frac{1}{1-X}$

Generating Function:

$$\sum_{k=0}^{\infty} a_k \cdot X^k = \frac{1-X}{1-X-2X^2} \ ,$$

Closed Form:

$$a_k = \frac{2^k + 2\cdot(-1)^k}{3}$$

Sequences of quotients' degrees

Second part: Characterize the *degrees* of the quotients

Example: $n = 4$, $\{1, x, x^2, x, 1\}$

$$(1,0) \xrightarrow{1} (1,1) \xrightarrow{x} (x+1,1) \xrightarrow{x^2} (x^3+x^2+1, x+1) \xrightarrow{x}$$
$$(x^4+x^3+1, x^3+x^2+1) \xrightarrow{1} (x^4+x^2+1, x^4+x^3+1)$$

Sum of degrees: $1+2+1 = 4$, $k = 3$

Question: what are the combinations of *ordered sums* of n?

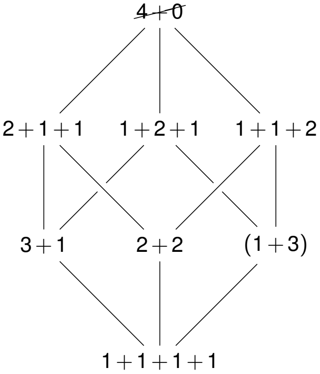
$\Rightarrow$ **compositions** of $n \in \mathbb{N}$

Quotients' degrees as compositions of n

- **Representation:** $n-1$ *boxes* that can be either "+" or ","

$$1\overbrace{\square 1 \square \ldots \square 1 \square}^{n-1} 1$$

- **Example:** $1, 1+1, 1 \rightarrow 1+2+1 \quad (n=4,\ k=3)$



- We remove the top of the poset
- **Enumeration**: generate all binary strings of length $1 < k < n$
- **Counting**: $\binom{n-1}{k-1}$

Enumeration Algorithm

- **Third part**: middle terms are *free*
- once k is fixed, all three parts are *independent*

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So for **enumeration**, given $n \in \mathbb{N}$:

For each composition *comp* of n of length k (except $k = 0$) do:

- Generate all quotients' sequences of *comp* (2^{n-k})
- For each quotients' sequence *seq* do:
  - For each constant term sequence of length k do:
    - Add the constant terms to the quotients
    - Apply DilcuE's from $(1,0)$ by applying *seq*

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$$\sum_{k=2}^{n} \underbrace{2^{n-k}}_{\text{middle}} \cdot \underbrace{\binom{n-1}{k-1}}_{\textit{degrees}} \cdot \underbrace{\frac{2^k + 2 \cdot (-1)^k}{3}}_{\textit{constant}}$$

Conclusions and Future Work

Summing up:

- Enumeration of binary coprime polynomials is more complicated when both constant terms are nonzero
- We divided the problem in three enumeration tasks:
  - sequences of constant terms ($\Rightarrow$ regular language)
  - sequences of degrees ($\Rightarrow$ compositions)
  - sequences of middle terms ($\Rightarrow$ free)

Future directions:

- Generalize to polynomials over any finite field $\mathbb{F}_q$
- Generalize to *m*-tuples of pairwise coprime polynomials
- Applications to cryptography and coding theory [GMP20, GM20, M21]

Thank you!

Appendix: Orthogonal Latin Squares (OLS)

Definition

A *Latin square* is a $n \times n$ matrix where all rows and columns are permutations of $[n] = \{1, \cdots, n\}$. Two Latin squares are *orthogonal* if their superposition yields all the pairs $(x, y) \in [n] \times [n]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

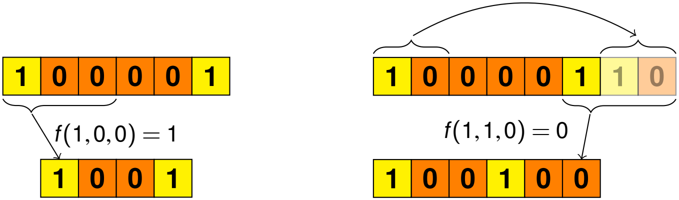
1, 1	3, 4	4, 2	2, 3
4, 3	2, 2	1, 4	3, 1
2, 4	4, 1	3, 3	1, 2
3, 2	1, 3	2, 1	4, 4

- k pairwise OLS are denoted as k-MOLS (**Mutually Orthogonal Latin Squares**)
- k-MOLS are **equivalent** $OA(n^2, k, n, 2)$

Appendix: Cellular Automata

- One-dimensional Cellular Automaton (CA): a discrete parallel computation model composed of a finite array of n cells

Example: $n = 6$, $d = 3$, $\omega = 0$, $f(s_i, s_{i+1}, s_{i+2}) = s_i \oplus s_{i+1} \oplus s_{i+2}$ (rule 150)



No Boundary CA – NBCA

Periodic Boundary CA – PBCA

- Each cell updates its state $s \in \{0,1\}$ by applying a local rule $f : \{0,1\}^d \to \{0,1\}$ to itself, the ω cells on its left and the $d - 1 - \omega$ cells on its right

Latin Squares through Bipermutive CA (1/2)

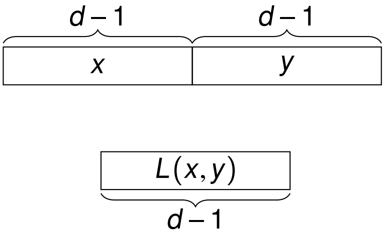
- Bipermutive CA: denoting $\mathbb{F}_2 = \{0,1\}$, local rule f is defined as

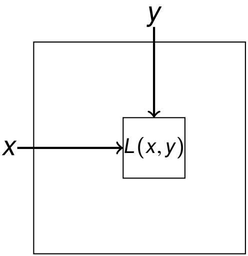
$$f(x_1,\cdots,x_d) = x_1 \oplus \varphi(x_2,\cdots,x_{d-1}) \oplus x_d$$

- $\varphi : \mathbb{F}_2^{d-2} \to \mathbb{F}_2$: generating function of f

Lemma ([MGFL20])

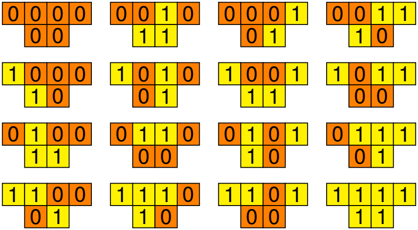
A CA $F : \mathbb{F}_2^{2(d-1)} \to \mathbb{F}_2^d$ with bipermutive rule $f : \mathbb{F}_2^d \to \mathbb{F}_2$ generates a Latin square of order $N = 2^{d-1}$





Latin Squares through Bipermutive CA (2/2)

- Example: CA $F : \mathbb{F}_2^4 \to \mathbb{F}_2^2$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

Mutually Orthogonal Cellular Automata (MOCA): set of k bipermutive CA generating k-MOLS

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