A new construction for bent functions based on cellular automata, Latin squares and linear recurring sequences

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Joint work with Maximilien Gadouleau and Stjepan Picek

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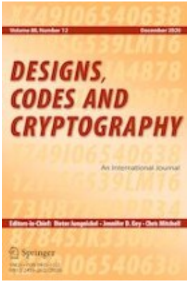
Introduction: an (unlucky) paper

M. Gadouleau, L. Mariot, S. Picek. Bent functions in the partial spread class generated by linear recurring sequences. Des. Codes and Cryptogr. 91:63–82 (2023)



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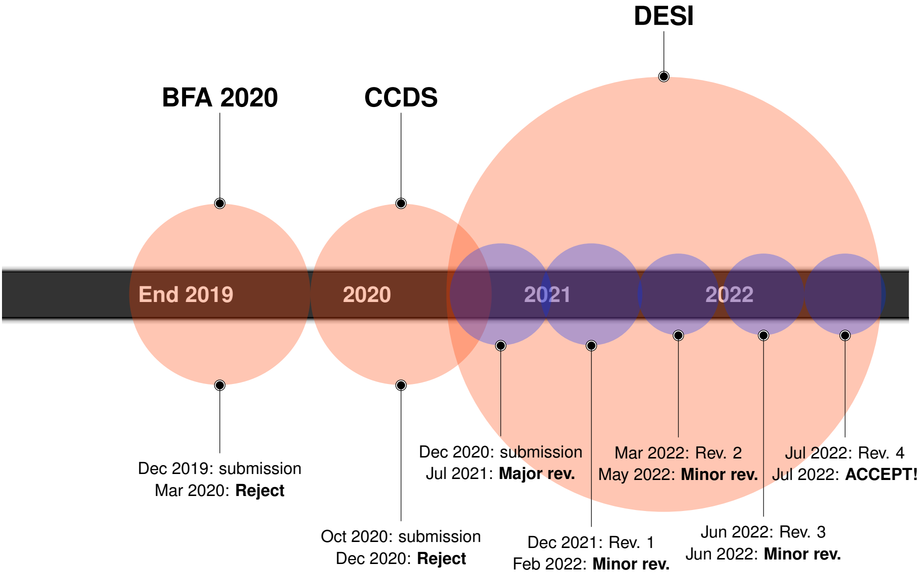
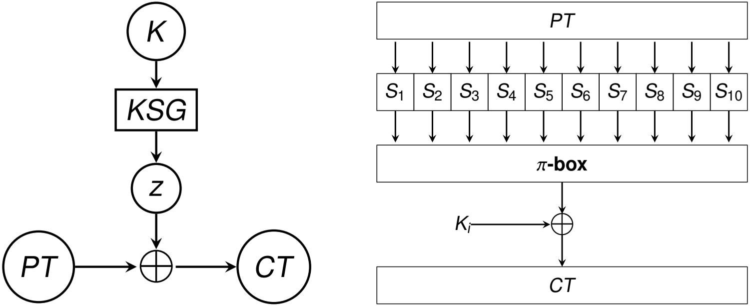
Peer-review Timeline



Boolean Functions in Symmetric Ciphers



(a) Stream cipher (b) Block cipher

Boolean functions $f : \{0,1\}^n \to \{0,1\}$ are used in [C21]

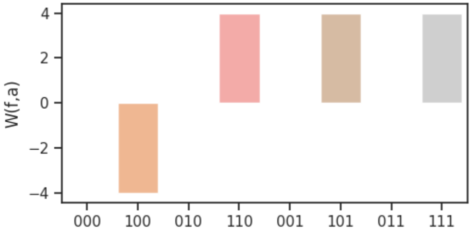
- **Stream ciphers**, to design the *keystream generator* (KSG)
- **Block ciphers**, as the coordinate functions of *S-boxes* (S_i)

Boolean Functions - Basic Representations

- Truth table: a 2^n-bit vector Ω_f specifying $f(x)$ for all $x \in \{0,1\}^n$

(x_1, x_2, x_3)	000	100	010	110	001	101	011	111
Ω_f	0	1	1	0	1	0	1	0

- Walsh Transform: correlation with linear functions $a \cdot x$,
  $W(f,a) = \sum_{x\in\{0,1\}^n} (-1)^{f(x)\oplus a\cdot x}$ for all $a \in \{0,1\}^n$

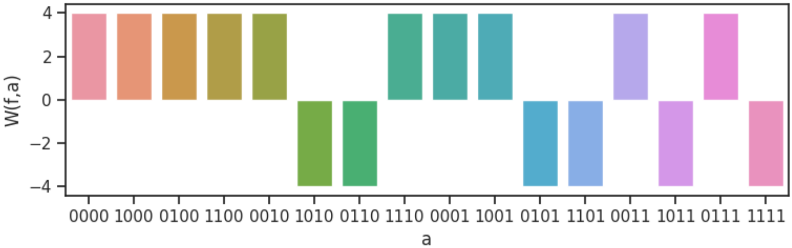


Bent Functions

- *Parseval's Relation*, valid on any Boolean function:

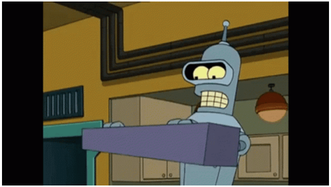
$$\sum_{a \in \{0,1\}^n} [W(f,a)]^2 = 2^{2n} \quad \text{for all} \quad f : \{0,1\}^n \to \{0,1\}$$

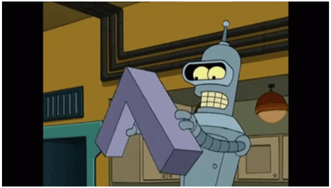
- Bent functions: $W(f,a) = \pm 2^{\frac{n}{2}}$ for all $a \in \{0,1\}^n$
  - Reach the highest possible *nonlinearity*
  - Exist only for n even and they are *unbalanced*



Example: $f(x_1, x_2, x_3, x_4) = x_1 x_3 + x_1 x_4 + x_2 x_4$

Intuition behind the name "bent"





- **Nonlinearity** of f: minimum Hamming distance of the truth table of f from all linear functions
- "Bent" functions are the farthest from linear ("straight") ones
- Related to the covering radius of **Reed-Muller codes**

Constructions of Bent Functions

Given $n = 2m$:

- **Maiorana-McFarland** [M73]): $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is defined as

$$f(x,y) = x \cdot \pi(y) \oplus g(y)$$

  where:
  - $\pi : \mathbb{F}_2^m \to \mathbb{F}_2^m$ permutation of $\mathbb{F}_2^m$
  - $g : \mathbb{F}_2^m \to \mathbb{F}_2$ any m-variable Boolean function

- **Partial spreads** [D74]: $f \in \mathcal{PS}^-$ ($f \in \mathcal{PS}^+$) is defined as

$$supp(f) = \bigcup_{S \in \mathcal{S}} (S \setminus \{\underline{0}\}) \ \left(supp(f) = \bigcup_{S \in \mathcal{S}} S \right) ,$$

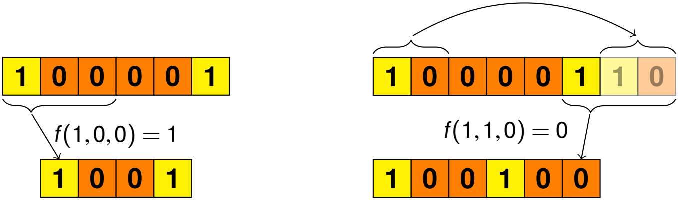
  with $\mathcal{S}$ a family of 2^{m-1} $(+1)$ m-dimensional subspaces of $\mathbb{F}_2^n$ with pairwise trivial intersection

Part 1: Cellular Automata and Mutually Orthogonal Latin Squares

Cellular Automata

- Vectorial functions $F : \mathbb{F}_q^n \to \mathbb{F}_q^m$ with *uniform* (shift-invariant) coordinates

Example: $q = 2$, $n = 6$, $d = 3$, $f(s_i, s_{i+1}, s_{i+2}) = s_i \oplus s_{i+1} \oplus s_{i+2}$



No Boundary CA – NBCA   Periodic Boundary CA – PBCA

- Concrete example: the χ transformation in KECCAK [BDPV11]:

$$q = 2, \;\; d = 3, \;\; \chi(x_i, x_{i+1}, x_{i+2}) = x_i \oplus (1 \oplus x_{i+1}) x_{i+2}$$

Mutually Orthogonal Latin Squares (MOLS)

Definition

A *Latin square* is a $n \times n$ matrix where all rows and columns are permutations of $[n] = \{1, \cdots, n\}$. Two Latin squares are *orthogonal* if their superposition yields all the pairs $(x, y) \in [n] \times [n]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

1 1	3 4	4 2	2 3
4 3	2 2	1 4	3 1
2 4	4 1	3 3	1 2
3 2	1 3	2 1	4 4

- k-**MOLS**: set of k pairwise orthogonal Latin squares

Latin Squares through Bipermutive CA (1/2)

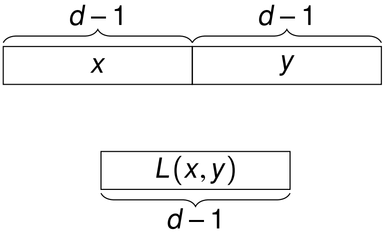
- Bipermutive CA: local rule f is defined as

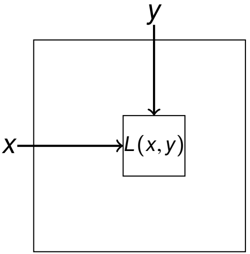
$$f(x_1, \cdots, x_d) = x_1 + \varphi(x_2, \cdots, x_{d-1}) + x_d$$

- $\varphi : \mathbb{F}_q^{d-2} \to \mathbb{F}_q$: generating function of f

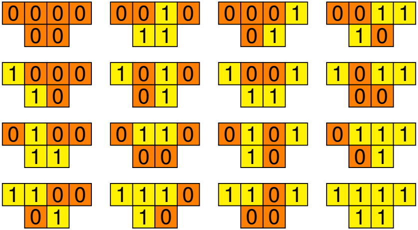
Lemma ([M16])

A (no-boundary) CA $F : \mathbb{F}_q^{2(d-1)} \to \mathbb{F}_q^d$ with bipermutive rule $f : \mathbb{F}_q^d \to \mathbb{F}_q$ generates a Latin square of order $N = q^{d-1}$





- Example: CA $F : \mathbb{F}_2^4 \to \mathbb{F}_2^2$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

Linear CA

- Local rule: *linear combination* of the neighborhood cells

$$f(x_1, \cdots, x_d) = a_1 x_1 + \cdots + a_d x_d \ \ , \ a_i \in \mathbb{F}_q$$

- Associated polynomial:

$$f \mapsto p_f(X) = a_1 + a_2 X + \cdots + a_d X^{d-1}$$

- $(n-d+1) \times n$ transition matrix:

$$M_F = \begin{pmatrix} a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_1 & \cdots & a_d \end{pmatrix}, \quad x \mapsto M_F x^\top$$

- **Remark:** a linear rule is bipermutive iff $a_1, a_d \neq 0$

MOLS from Linear Bipermutive CA (LBCA)

Theorem ([MGLF20])

A set of t linear bipermutive CA $F_1, \ldots F_t : \mathbb{F}_q^{2(d-1)} \to \mathbb{F}_q^{d-1}$ generates a family of t-MOLS of order $N = q^{d-1}$ if and only if their associated polynomials are pairwise coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1, 1	4, 2	3, 3	2, 4
2, 2	3, 1	4, 4	1, 3
4, 3	1, 4	2, 1	3, 2
3, 4	2, 3	1, 2	4, 1

(c) Superposition

Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Part 2: The Complicated Construction

Hadamard Matrices

- Hadamard Matrix: a $n \times n$ matrix with ± 1 entries and s.t. $H \cdot H^\top = I_n$

$$H = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}, \; n = 4$$

- Necessary condition: $n = 1, 2$ or $n = 4k$
- Hadamard Conjecture: a Hadamard matrix exists for every $n = 4k$

Hadamard Matrices and Bent Functions

Theorem (Dillon, 1974 [D74])

Given $f : \{0,1\}^n \to \{0,1\}$ and $\hat{f}(x) = (-1)^{f(x)}$. Define the $2^n \times 2^n$ matrix H for all $x, y \in \{0,1\}^n$ as:

$$H(x,y) = \hat{f}(x \oplus y)$$

Then, f is a bent function if and only if H is a Hadamard matrix.

Example: $f(x_1, x_2) = x_1 x_2$

x_1	x_2	$x_1 x_2$
0	0	0
1	0	0
0	1	0
1	1	1

$$H = \begin{pmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{pmatrix}$$

Hadamard Matrices from MOLS

Orthogonal Array $OA(t,N)$ for t MOLS of order N: $N^2 \times (t+2)$ matrix where each Latin square is "linearized" as a column

L_{90} $(1+X^2)$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

L_{150} $(1+X+X^2)$

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

$\Rightarrow$

x	y	L_{90}	L_{150}
1	1	1	1
1	2	2	4
1	3	3	3
1	4	4	2
2	1	2	2
2	2	1	3
2	3	4	4
2	4	3	1
3	1	3	4
3	2	4	1
3	3	1	2
3	4	2	3
4	1	4	3
4	2	3	2
4	3	2	1
4	4	1	4

Theorem (Bush, 1973 [B73])

Given t MOLS of order $N = 2t$, there exists a $4t^2 \times 4t^2$ symmetric Hadamard matrix H

Construction:

- Put – only in (i,j) where $i \neq j$ and there is a column k in the OA s.t the rows i and j have the same symbol
- Put $+$ everywhere else

From Linear CA to Bent Functions

- **Question**: Are MOLS arising from linear CA suitable for constructing bent functions?
- We consider only CA over $\mathbb{F}_q$ with $q = 2^l$, $l \in \mathbb{N}$
- The order of the Hadamard matrix must be $4t^2 = 2^n$
- We need t coprime polynomials of degree $b = d - 1$:

$$2^{lb} = 2t \Leftrightarrow lb = 1 + \log_2 t$$

- Since both l and b are integers, $t = 2^w$ for $w \in \mathbb{N}$

From Linear CA to Bent Functions

Theorem

Let H be the Hadamard matrix of order $2^{2(w+1)}$ defined by the t LBCA $F_1, \cdots F_t : \mathbb{F}_q^{2b} \to \mathbb{F}_q^b$, and define $f : \mathbb{F}_2^n \to \mathbb{F}_2$, $n = 2(w+1)$ as:

$$f(x) = \begin{cases} 0 \ , & \text{if } x = 0 \\ 1 \ , & \text{if } x \neq 0 \text{ and } \exists k \in \{1, \cdots, t\} \text{ s.t. } F_k(x) = 0 \\ 0 \ , & \text{otherwise} \end{cases} ,$$

Then, it holds that:

$$H(x, y) = \hat{f}(x \oplus y)$$

and thus f is a bent function

Remark: The linearity of the CA is crucial to grant this result (and costed us our first reject!)

Example

$$p_f(X) = 1 + X^2$$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$$p_g(X) = 1 + X + X^2$$

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

$$A = \left\{ \begin{array}{|c|c|} \multicolumn{1}{c}{L_1} & \multicolumn{1}{c}{L_2} \\ \hline 1 & 1 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 2 & 2 \\ 1 & 3 \\ 4 & 4 \\ 3 & 1 \\ 3 & 4 \\ 4 & 1 \\ 1 & 2 \\ 2 & 3 \\ 4 & 3 \\ 3 & 2 \\ 2 & 1 \\ 1 & 4 \\ \hline \end{array} \right.$$

$$H = \begin{pmatrix}
+&+&+&+&+&-&-&+&+&+&-&-&+&-&+&- \\
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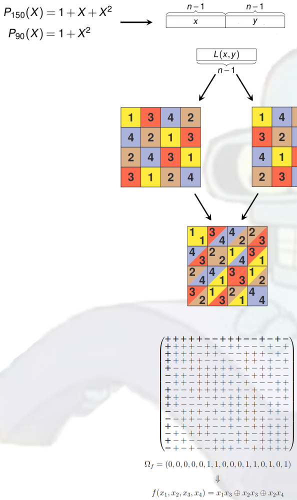
\end{pmatrix}$$

$$\Omega_f = (0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1)$$

$$\Downarrow$$

$$f(x_1, x_2, x_3, x_4) = x_1x_3 \oplus x_2x_3 \oplus x_2x_4$$

Figure 3: Example of bent function of $n = 4$ variables generated by the $t = 2$ MOLS of order $2t = 4$ defined by the LBCA with rule 90 and 150, respectively. The two Latin squares are represented on the left in the OA form. The first row and the first column of the Hadamard matrix H coincide with the polarity truth table of the function.



Combinatorial questions addressed in [GMP20]:

- **Existence:** for even n, does a large enough family of coprime polynomials exist?
- **Counting:** how many families of this kind exist (= number of CA-based bent functions)?

Part 3: A Simplified Construction with Linear Recurring Sequences

Reviewers Feedback

- First attempt: BFA, reject (incomplete proof)
- Second attempt: CCDS, reject (complicated construction, no guarantee the obtained bent functions are new)
- Third attempt: DESI, major revision

Strictest (and most enthusiastic!) review:

This paper must be published in some form! :)
It has the potential of becoming a major reference on bent functions because it identifies a new source of partial spreads large enough to give bent functions!

...

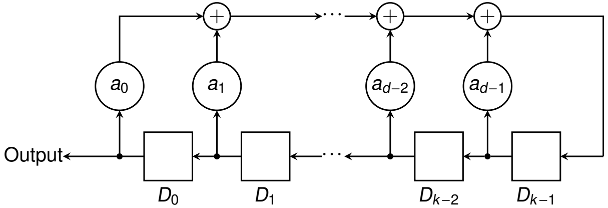
But not in its present form which buries and obscures their great new result in many pages of material on subjects which, in retrospect, are unnecessary for a lucid exposition of their new results. It may be in order to *briefly* mention how they were led to their theorem via the CA, Latin square, MOLS and Bush

Linear Recurring Sequences (LRS)

- Sequence $\{x_i\}_{i\in\mathbb{N}}$ satisfying the following relation:

$$a_0 x_i + a_1 x_{i+1} + ... + a_{d-1} x_{i+d-1} = x_{i+d}$$

- Computed by a *Linear Feedback Shift Register* (LFSR):



- Feedback polynomial:

$$f(X) = a_0 + a_1 X + \cdots a_{d-1} X^{d-1} + X^d$$

Linear map associated to a LRS

- Take the *projection* of all sequences satisfying the LRS defined by $f(X)$ onto their first $2d$ coordinates
- Obtain a d-dim subspace $S_f \subseteq \mathbb{F}_q^{2d}$ which is the kernel of the linear map $F : \mathbb{F}_q^{2d} \to \mathbb{F}_q^d$:

$$F(x_0, \cdots, x_{2d-1})_i = a_0 x_i + a_1 x_{i+1} + ... + a_{d-1} x_{i+d-1} + x_{i+d} \ ,$$

associated matrix:

$$M_F = \begin{pmatrix} a_0 & \cdots & a_{d-1} & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_0 & \cdots & a_{d-1} & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & a_0 & \cdots & a_{d-1} & 1 \end{pmatrix}$$

- ... but this is *exactly* the global rule of a linear CA!

Partial Spreads from Coprime Polynomials

Lemma

Given $f, g \in \mathbb{F}_q[X]$ over $\mathbb{F}_q$ of degree $d \geq 1$, defined as:

$$f(X) = a_0 + a_1 X + \cdots + a_{d-1} X^{d-1} + X^d \quad , \tag{1}$$

$$g(X) = b_0 + b_1 X + \cdots + b_{d-1} X^{d-1} + X^d \quad , \tag{2}$$

Then, the kernels of $F, G : \mathbb{F}_q^{2d} \to \mathbb{F}_q^d$ have trivial intersection if and only if $\gcd(f, g) = 1$

Consequence: a family of t pairwise coprime polynomials defines a partial spread

Equivalence check

For degree $b = 1$, actually nothing new:

Lemma

Our construction coincides with the class $\mathcal{PS}_{ap}$ when $b = 1$.

For degree $b = 2$:

- Computed the ranks of the associated Hadamard matrices in binary form to check equivalence
- **1st Finding**: none of our functions are equivalent to Maiorana-McFarland ones
- **2nd Finding**: many of our functions are not even equivalent to $\mathcal{PS}_{ap}$ ones

Conclusions

Recap and Open Problems

Remarkable findings:

- (Complicated!) construction of bent functions via CA, Latin Squares and Hadamard matrices
- Simplification based on kernels of LRS subspaces
- Resulting bent functions coincide with $\mathcal{PS}_{ap}$ for degree $b = 1$
- For $b = 2$, many functions are not in $\mathcal{PS}_{ap}$

Open problems:

- Are functions from polynomials of degree $b = 2$ *really* new?
- Implementation of CA-based bent functions via LFSR [ML18]

References

[BDPV11] Bertoni, G., Daemen, J., Peeters, M., Van Assche, G. 2011. The Keccak reference. http://keccak.noekeon.org/ (2011)

[B73] K. Bush: Construction of symmetric Hadamard matrices. In: A survey of combinatorial theory, pp. 81–83. Elsevier (1973)

[C21] C. Carlet: Boolean functions for cryptography and coding theory. Cambridge University Press (2021)

[D74] J.F. Dillon.: Elementary Hadamard difference sets. Ph.D. thesis (1974)

[GMP23] M. Gadouleau, L. Mariot, S. Picek. Bent functions in the partial spread class generated by linear recurring sequences. Des. Codes and Cryptogr. (2022) DOI: https://doi.org/10.1007/s10623-022-01097-1

[GMP20] M. Gadouleau, L. Mariot, S. Picek: Bent Functions from Cellular Automata. IACR Cryptol. ePrint Arch. 2020: 1272 (2020)

[MGLF20] L. Mariot, M. Gadouleau, E. Formenti, and A. Leporati. Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)

[M16] L. Mariot, E. Formenti, A. Leporati: Constructing Orthogonal Latin Squares from Linear Cellular Automata. In: Exploratory papers of AUTOMATA 2016 (2016)

[ML18] L. Mariot, A. Leporati: A cryptographic and coding-theoretic perspective on the global rules of cellular automata. Nat. Comput. 17(3):487–498 (2018)

[M73] R. L. McFarland. A family of difference sets in non-cyclic groups. J. Comb. Theory, Ser. A 15(1):1–10 (1973)

[M16] S. Mesnager: Bent Functions – Fundamentals and Results. Springer (2016)

[W07] G. Weng, R. Feng, W. Qiu: On the ranks of bent functions. Finite Fields Their Appl. 13(4), 1096–1116 (2007)