Sharing Secrets by Computing Preimages of Bipermutive CA

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One-Dimensional Cellular Automata

Definition

A finite boolean one-dimensional cellular automaton (CA) is a triple $\langle n, r, f \rangle$ where $n \in \mathbb{N}$ is the number of cells, $r \in \mathbb{N}$ is the radius and $f : \mathbb{F}_2^{2r+1} \rightarrow \mathbb{F}_2$ is a boolean function specifying the CA local rule.

- During a single time step, a cell i updates its boolean state c_i in parallel by computing $f(c_{i-r}, \cdots, c_i, \cdots, c_{i+r})$
- No Boundary CA: only the central cells $i \in \{r+1, \cdots, n-r\}$ update their states; the array shrinks by $2r$ cells at each time step

Secret Sharing Schemes: Basic Definitions

- A secret sharing scheme is a procedure which enables a dealer to share a secret S among a set $\mathcal{P}$ of players, in such a way that only some authorized subsets can recover S
- An access structure $\Gamma \subseteq 2^{\mathcal{P}}$ specifies the authorized subsets
- In (k, n) threshold schemes, the access structure Γ contains all those subsets of at least k players
- Shamir's scheme [Shamir79], which is based on polynomial interpolation, is an example of (k, n) threshold scheme
- The CA-based scheme proposed in [Rey05] features a sequential (k, n) threshold scheme

Perfect and Ideal Secret Sharing Schemes

- Let us assume that a probability distribution $Pr(S)$ is defined on the space of the secrets, and that δ_U represents a shares distribution to an unauthorized subset $U \notin \Gamma$
- A secret sharing scheme is perfect if for all unauthorized subsets $U \notin \Gamma$ and for all shares distributions δ_U it results that

$$Pr(S|\delta_U) = Pr(S)$$

- A secret sharing scheme is called ideal if the size of each share equals the size of the secret

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Permutive and Bipermutive Rules

Rule $f : \mathbb{F}_2^{2r+1} \rightarrow \mathbb{F}_2$ is called:

- leftmost permutive if there exists $g_L : \mathbb{F}_2^{2r} \rightarrow \mathbb{F}_2$ such that:

$$f(x_1, x_2, \cdots x_{2r+1}) = x_1 \oplus g_L(x_2, \cdots, x_{2r+1})$$

- rightmost permutive if there exists $g_R : \mathbb{F}_2^{2r} \rightarrow \mathbb{F}_2$ such that:

$$f(x_1, \cdots, x_{2r}, x_{2r+1}) = g_R(x_1, \cdots, x_{2r}) \oplus x_{2r+1}$$

- bipermutive if there exists $g : \mathbb{F}_2^{2r-1} \rightarrow \mathbb{F}_2$ such that:

$$f(x_1, x_2, \cdots, x_{2r}, x_{2r+1}) = x_1 \oplus g(x_2, \cdots, x_{2r}) \oplus x_{2r+1}$$

Building Preimages of (Bi)Permutive CAs [Gutowitz93] (1/6)

Given a rightmost permutive rule $f : \mathbb{F}_2^{2r+1} \rightarrow \mathbb{F}_2$ and a configuration $c \in \mathbb{F}_2^m$, a preimage $p \in \mathbb{F}_2^{m+2r}$ of c can be computed as follows:

1. Set the leftmost $2r$ cells $p_1, \cdots, p_{2r}$ of the preimage p to random values

p =	**0**	**1**	**?**	**?**	**?**	**?**	**?**	**?**

c =	**1**	**0**	**0**	**1**	**1**	**0**

Figure: Example of preimage construction under rule 30 (R-permutive)

Building Preimages of (Bi)Permutive CAs [Gutowitz93] (2/6)

Given a rightmost permutive rule $f : \mathbb{F}_2^{2r+1} \to \mathbb{F}_2$ and a configuration $c \in \mathbb{F}_2^m$, a preimage $p \in \mathbb{F}_2^{m+2r}$ of c can be computed as follows:

2. By right permutivity, $c_1 = g_R(p_1, \cdots, p_{2r}) \oplus p_{2r+1}$. Hence, p_{2r+1} can be computed as $p_{2r+1} = g_R(p_1, \cdots, p_{2r}) \oplus c_1$

p =	**0**	**1**	**?**	**?**	**?**	**?**	**?**	**?**
c =		**1**	**0**	**0**	**1**	**1**	**0**	

Figure: Example of preimage construction under rule 30 (R-permutive)

Building Preimages of (Bi)Permutive CAs [Gutowitz93] (3/6)

Given a rightmost permutive rule $f : \mathbb{F}_2^{2r+1} \to \mathbb{F}_2$ and a configuration $c \in \mathbb{F}_2^m$, a preimage $p \in \mathbb{F}_2^{m+2r}$ of c can be computed as follows:

3. Shift the $2r$-bit window one place to the right and compute $p_{2r+2} = g_R(p_2, \cdots, p_{2r+1}) \oplus c_2$

p =	**0**	**1**	**0**	**?**	**?**	**?**	**?**	**?**
c =		**1**	**0**	**0**	**1**	**1**	**0**	

Figure: Example of preimage construction under rule 30 (R-permutive)

Building Preimages of (Bi)Permutive CAs [Gutowitz93] (4/6)

Given a rightmost permutive rule $f : \mathbb{F}_2^{2r+1} \to \mathbb{F}_2$ and a configuration $c \in \mathbb{F}_2^m$, a preimage $p \in \mathbb{F}_2^{m+2r}$ of c can be computed as follows:

4. Continue to apply Step 3 until the rightmost bit in the preimage has been computed

$p =$	0	1	0	1	?	?	?	?
$c =$		1	0	0	1	1	0	

Figure: Example of preimage construction under rule 30 (R-permutive)

Building Preimages of (Bi)Permutive CAs [Gutowitz93] (5/6)

Given a rightmost permutive rule $f : \mathbb{F}_2^{2r+1} \to \mathbb{F}_2$ and a configuration $c \in \mathbb{F}_2^m$, a preimage $p \in \mathbb{F}_2^{m+2r}$ of c can be computed as follows:

4. Continue to apply Step 3 until the rightmost bit in the preimage has been computed

$p =$	0	1	0	1	1	0	0	0
$c =$		1	0	0	1	1	0	

Figure: Example of preimage construction under rule 30 (R-permutive)

Building Preimages of (Bi)Permutive CAs [Gutowitz93] (6/6)

- For leftmost permutive rules, a symmetrical result holds by starting from the right and completing leftwards
- Each image in a rightmost (leftmost) permutive CA has thus 2^{2r} preimages
- If f is bipermutive, the initial block can be placed at any position [Oliveira04]. This possibility does not increase the number of preimages

$p =$

?	?	?	?	0	1	?	?

$c =$

1	0	0	1	1	0

(a) Initialization

$p =$

1	0	0	0	0	1	0	1

$c =$

1	0	0	1	1	0

(b) Complete preimage

Figure: Example with bipermutive rule 150

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Observations on Preimage Computation

- By iterating the procedure of preimage computation, at each step the size of the preimage grows by $2r$ cells
- In particular, starting from a CA configuration c of length m, after t steps the resulting preimage will have length $L(t) = 2rt + m$
- Hence, given $k \in \mathbb{N}$, the number of iterations t necessary to get a preimage of length $k \cdot m$ is:

$$t = \frac{m(k-1)}{2r}$$

- Since t is integer, it means that $2r$ must divide $m(k-1)$
- Additional security requirement: $2r|m$

Basic (k, k) Secret Sharing Scheme - Setup Phase (1/5)

Assuming that there are k players $P_1, P_2, \cdots, P_k$:

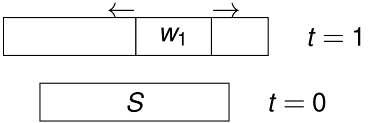
1. The *dealer D* sets the secret S as an m-bit configuration of a CA, and randomly selects a bipermutive rule of radius r, where r is such that $2r \mid m$

$\boxed{\quad S \quad}$ $t = 0$

Basic (k, k) Secret Sharing Scheme - Setup Phase (2/5)

Assuming that there are k players $P_1, P_2, \cdots, P_k$:

2. D evolves the CA backwards for $T = m(k-1)/2r$ iterations, randomly choosing at each step the value and the position of the initial $2r$-bit block



Basic (k, k) Secret Sharing Scheme - Setup Phase (3/5)

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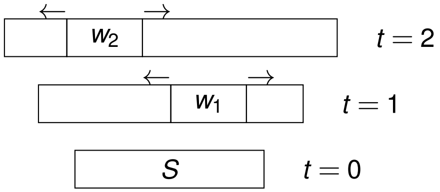
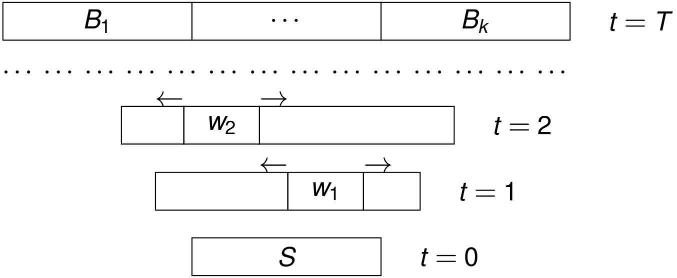


Basic (k, k) Secret Sharing Scheme - Setup Phase (4/5)

Assuming that there are k players $P_1, P_2, \cdots, P_k$:

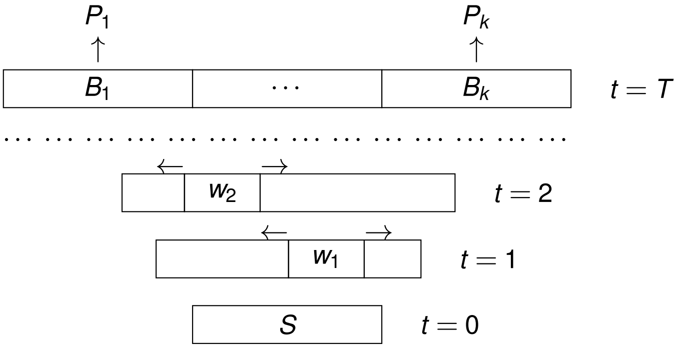
3. After $T = m(k-1)/2r$ iterations, the dealer splits the resulting preimage in k blocks of m bits



Basic (k, k) Secret Sharing Scheme - Setup Phase (5/5)

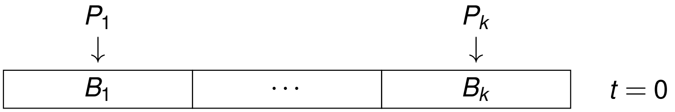
Assuming that there are k players $P_1, P_2, \cdots, P_k$:

4. Finally, D securely sends one block to each player and publishes the bipermutive rule used to evolve the CA backwards



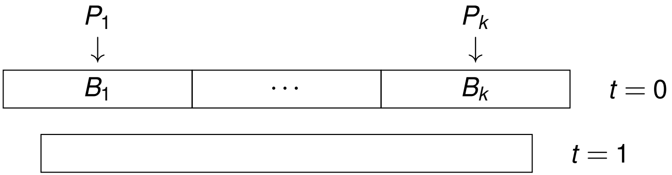
Basic (k, k) Secret Sharing Scheme - Recovery Phase (1/4)

1. Using a pre-established protocol, the k players pool their shares in the correct order to get the complete preimage of the CA



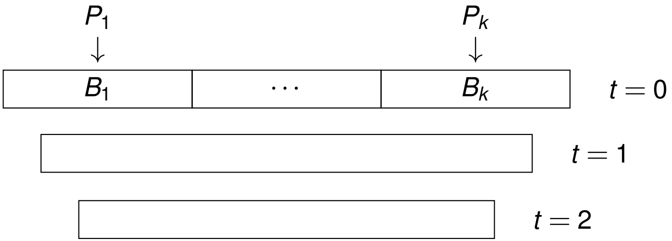
Basic (k, k) Secret Sharing Scheme - Recovery Phase (2/4)

2. The players evolve the CA forward, using the local rule published by the dealer



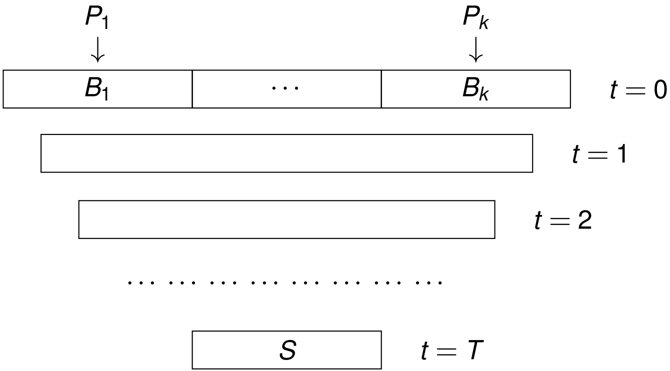
Basic (k, k) Secret Sharing Scheme - Recovery Phase (3/4)

2. The players evolve the CA forward, using the local rule published by the dealer



Basic (k,k) Secret Sharing Scheme - Recovery Phase (4/4)

3. The configuration obtained after $T = m(k-1)/2r$ iterations is the secret S. Notice that the players can compute T by themselves



Security Properties of the Basic Scheme

Lemma

Let B_l, with $1 \leq l \leq k$, be the only unknown share among $B_1, \cdots, B_k$. Then, under the condition that $2r \mid m$, there exists a permutation $\Pi : \mathbb{F}_2^m \to \mathbb{F}_2^m$ between B_l and the secret S.

From the previous Lemma, the following result holds:

Theorem

Suppose that the secret S and the $2r$-bit blocks in the setup phase are chosen uniformly at random. Then, the basic (k, k) scheme is perfect

Moreover, the basic scheme is also ideal, since each share is a block of m bits, as the secret

Outline

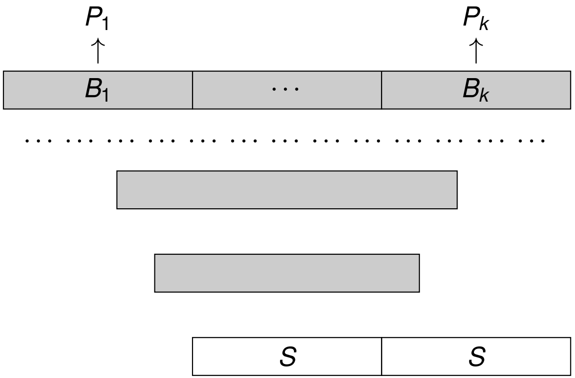
- Cellular Automata and Secret Sharing Schemes
- Building Preimages of Bipermutive CAs
- A New (k, k) Scheme Based on Bipermutive CAs
- **An Extension to the Basic Scheme**
- Conclusions and Future Developments

Considerations on the Basic scheme

- The basic scheme can be used to implement any access structure $\Gamma \subseteq 2^{\mathcal{P}}$: simply re-run the setup phase for each authorized subset $A \in \Gamma$
- However, as the number of participants grows, the scheme turns out to be impractical, since each player must hold a different share for each authorized subset he belongs to
- Necessity to find an extended scheme which allows the players to reuse the same shares
- Suppose that a set of k shares has been distributed to k players using the basic setup phase. The scheme can be extended using secret juxtaposition

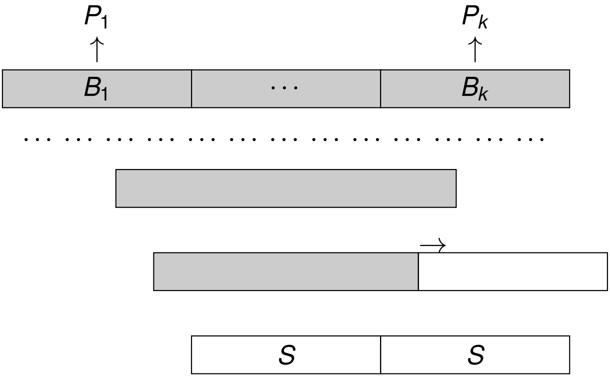
Secret Juxtaposition (1/4)

1. Append a copy of the secret S to the right of the final CA image



Secret Juxtaposition (2/4)

2. Update the preimages by completing them rightwards (note that it is not necessary to pick extra random bits)



Secret Juxtaposition (3/4)

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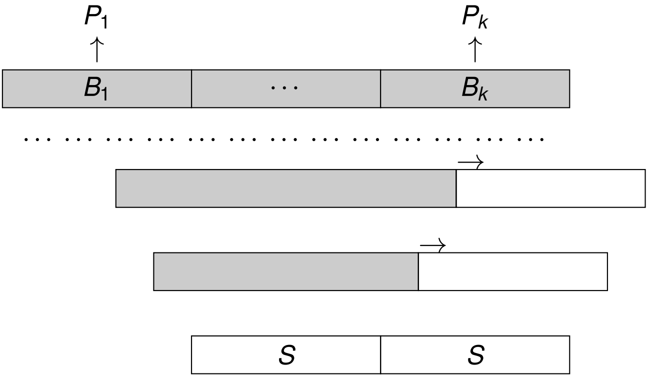
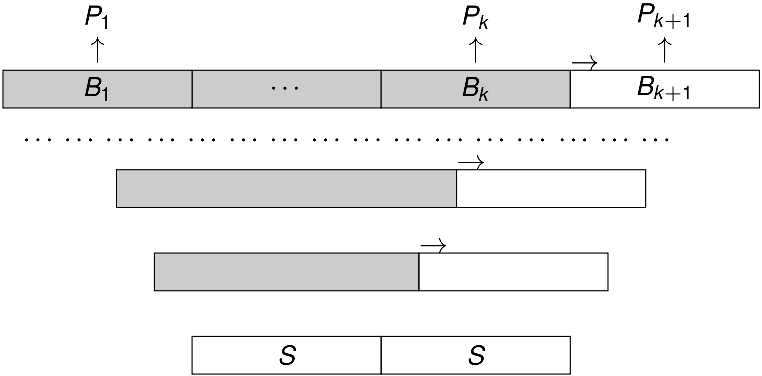


Secret Juxtaposition (4/4)

3. The last preimage contains an additional block for the new player. The sets $\{P_1, \cdots, P_k\}$ and $\{P_2, \cdots, P_{k+1}\}$ can recover S



Access Structure of the Extended Scheme

- The extended scheme implements a (k, n)-sequential threshold access structure: at least k consecutive shares are necessary to recover the secret
- In particular, if we continue to append copies of the secret, the final shares will eventually repeat. Thus, the access structure becomes cyclic

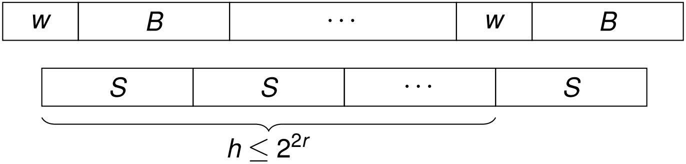


Figure: After at most $h \leq 2^{2r}$ juxtaposed copies of S, by completing rightwards the $2r$-bit block w will repeat at the end of the preimage.

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Conclusions

- We showed how the surjectivity of bipermutve CAs can be employed to design a basic secret sharing scheme where all the players are required in order to recover the secret S
- This basic scheme can be proved to be both perfect and ideal
- The secret juxtaposition method allows to extend the basic scheme with a cyclic access structure

Future Developments

- Find a general method to compute after how many juxtapositions of the secret the shares begin to repeat themselves. This is equivalent to the following open problem:

Open Problem (PCAP - Periods of CA Preimages)

Given a bipermutive CA and a spatially periodic configuration $c \in A^{\mathbb{Z}}$ with period m, find the periods of its preimages

- Recent investigation indicates that PCAP can be completely solved in the case of additive bipermutive CAs
- Other improvements: investigate possible applications of the scheme to secure multiparty computation protocols, and extend the scheme to d-dimensional CAs with $d > 1$

Thanks for your attention!

References

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