

Enumerating Orthogonal Latin Squares Generated by Bipermutive Cellular Automata

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One-Dimensional Cellular Automata (CA)

Definition

One-dimensional CA: triple $\langle m, n, f \rangle$ where $m \in \mathbb{N}$ is the number of cells on a one-dimensional array, $n \in \mathbb{N}$ is the neighborhood and $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is the local rule.

Example: $m = 8, n = 3, f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)

...

0	1	1	0	0
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 ...

$f(1, 1, 0) = 1 \oplus 1 \oplus 0$

0

1	0	0	0	0	1	0	1
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Parallel update \Downarrow Global rule F

1	0	0	1	1	0
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CA Global Rule: $F : \{0, 1\}^m \rightarrow \{0, 1\}^{m-n+1}$ defined as

$$F(x_1, \dots, x_m) = (f(x_1, \dots, x_n), f(x_2, \dots, x_{n+1}), \dots, f(x_{m-n+1}, \dots, x_m))$$

Latin Squares and Quasigroups

Definition

Latin square of order N : a $N \times N$ matrix L such that every row and every column are permutations of $[N] = \{1, \dots, N\}$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Latin square of order N



Cayley table of quasigroup
 (Q, \circ) with $|Q| = N$

Definition

Quasigroup: algebraic structure (Q, \circ) where for all $x, y \in Q$ the equations $x \circ z = y$ and $z \circ x = y$ have a unique solution for $z \in Q$

Orthogonal Latin Squares

Definition

Two Latin squares L_1 and L_2 of order n are *orthogonal* if their superposition yields all the pairs $(x, y) \in [N] \times [N]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

(a) L_1

1	4	2	3
3	2	4	1
4	1	3	2
2	3	4	1

(b) L_2

1,1	3,4	4,2	2,3
4,3	2,2	1,4	3,1
2,4	4,1	3,3	1,2
3,2	1,3	2,1	4,4

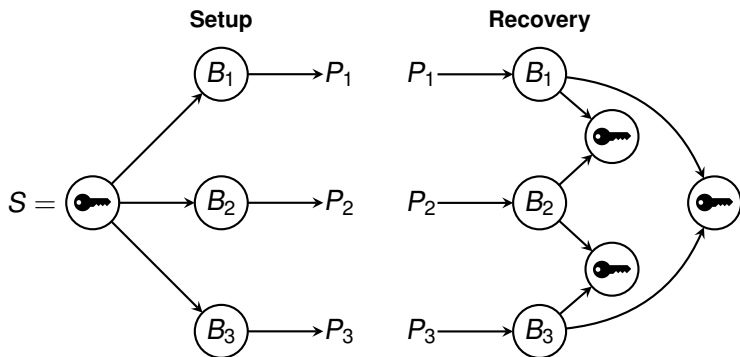
(c) (L_1, L_2)

A set of n pairwise orthogonal Latin squares is denoted as n -MOLS

Secret Sharing Schemes (SSS)

(k, n) **Threshold Secret Sharing Scheme**: a procedure enabling a **dealer** to share a **secret** S among n **players** so that at least k players out of n can recover S [Shamir79].

Example: $(2, 3)$ -scheme



Remark: $(2, n)$ -scheme \Leftrightarrow set of n -MOLS

SSS based on Cellular Automata: Why?

Twofold motivation:

- ▶ **Theoretical**: access structures arising from SSS where CA are used in a “natural” and simple way
- ▶ **Practical**: CA-based threshold schemes \Rightarrow Efficient (parallel) implementation of threshold schemes

Remark: All the published CA-based SSS [Mariot14, DelRey05] provide a **sequential** threshold access structure (the shares need to be adjacent)

First Question: Can (k, n) -schemes be realised through CA?

Latin Squares through Bipermutive CA (1/2)

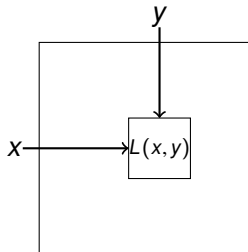
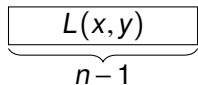
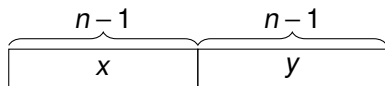
- ▶ **Idea:** determine which CA induce orthogonal Latin squares
- ▶ **Bipermutive CA:** local rule f is defined as

$$f(x_1, \dots, x_n) = x_1 \oplus \varphi(x_2, \dots, x_{2r}) \oplus x_n$$

- ▶ $\varphi : \{0, 1\}^{n-2} \rightarrow \{0, 1\}$: **generating function** of f

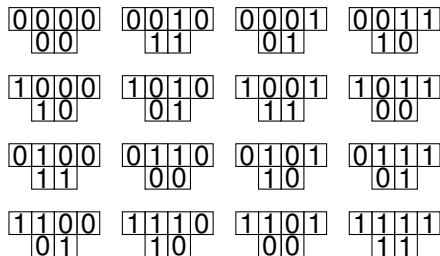
Lemma ([Eloranta93, Mariot16])

Let $\langle 2(n-1), n, f \rangle$ be a CA with bipermutive rule. Then, the global rule F generates a Latin square of order $N = 2^{n-1}$



Latin Squares through Bipermutive CA (2/2)

- ▶ **Example:** CA $\langle 4, 1, f \rangle$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- ▶ Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

Orthogonal Latin Squares by Linear CA

- ▶ **Bipermutive Linear rule:** $f(x) = x_1 \oplus a_2 x_2 \oplus \dots \oplus a_{n-1} x_{n-1} \oplus x_n$
- ▶ **Associated polynomial:** $f \mapsto P_f(X) = a_1 + a_2 X + \dots + a_n X^{n-1}$

Theorem ([Mariot16])

Bipermutive linear rules $f, g : \{0, 1\}^n \rightarrow \{0, 1\}$ generate orthogonal Latin squares if and only if $P_f(X)$ and $P_g(X)$ are coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1,1	4,2	3,3	2,4
2,2	3,1	4,4	1,3
4,3	1,4	2,1	3,2
3,4	2,3	1,2	4,1

(c) Superposition

Figure : $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Enumerating CA-based OLS

- ▶ Enumeration of OLS in the linear case \Leftrightarrow Enumeration of pairs of coprime polynomials (But that's another story...)
- ▶ ... What about the **nonlinear** case?
- ▶ MOLS arising from nonlinear constructions have relevance in **cheater-immune** Secret Sharing Schemes [Tomba88]

Goal: Exhaustive enumeration of pairs of bipermutive rules of size n generating orthogonal Latin squares, classified by nonlinearity

Nonlinearity

- ▶ **Affine function**: $l(x_1, \dots, x_n) = a \oplus a_1 x_1 \oplus \dots \oplus a_n x_n$, $a, a_i \in \{0, 1\}$
- ▶ **Nonlinearity** of f : Hamming distance of the truth table of f from the set of all affine functions
- ▶ **Walsh transform** of f : given $\omega \in \{0, 1\}^n$,

$$W_f(\omega) = \sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus \omega \cdot x}, \text{ where } \omega \cdot x = \bigoplus_{i=1}^n \omega_i \cdot x_i$$

Definition

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$. The *nonlinearity* of f is defined as

$$NI(f) = 2^{n-1} - \frac{1}{2} \max_{\omega \in \{0,1\}^n} \{|W_f(\omega)|\}$$

Search Space Size

- ▶ Number of Boolean functions of n variables: $\mathcal{F}_n = 2^{2^n}$
- ▶ Bipermutive rules of size $n \Leftrightarrow$ Generating functions of size $n-2$ (which are $\mathcal{F}_{n-2} = 2^{2^{n-2}}$)
- ▶ Pairs of bipermutive rules of size n : $\mathcal{B}_n = 2^{2^{n-1}} = \mathcal{F}_{n-1}$

n	3	4	5	6	7
\mathcal{B}_n	16	256	65536	4294967296	$\approx 1.84 \cdot 10^{19}$

- ▶ **Remark:** Exhaustive enumeration possible up to $n = 6$
- ▶ How can we further prune the search space?

- ▶ **Reversal** of f : $f_R(x_1, \dots, x_n) = f(x_n, \dots, x_1)$
- ▶ **Complement** of f : $f_C(x_1, \dots, x_n) = 1 \oplus f(x_1, \dots, x_n)$

Lemma

Let $(f, g) : \{0, 1\}^n \rightarrow \{0, 1\}$ be bipermutive rules generating orthogonal Latin squares. Then, the Latin squares respectively induced by (f_R, g_R) and (f_C, g_C) are orthogonal as well

- ▶ Clearly, the swapped pair (g, f) generates the orthogonal Latin squares in swapped order
- ▶ Hence, the search space can be divided by 8

Pairwise Balancedness (PWB)

Definition

$f, g : \{0, 1\}^n \rightarrow \{0, 1\}$ are **pairwise balanced** (PWB) if

$$\begin{aligned} |(f, g)^{-1}(0, 0)| &= |(f, g)^{-1}(1, 0)| = \\ &= |(f, g)^{-1}(0, 1)| = |(f, g)^{-1}(1, 1)| = 2^{n-2} \end{aligned}$$

Example:

- ▶ $f(x_1, x_2, x_3) = x_1 \oplus x_3$ (Rule 90)
- ▶ $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)

$$\Omega(f) = (0, 1, 0, 1, 1, 0, 1, 0) ,$$

$$\Omega(g) = (0, 1, 1, 0, 1, 0, 0, 1) .$$

Each of the pairs $(0, 0), (1, 0), (0, 1), (1, 1)$ occurs $2^{3-2} = 2$ times

Lemma

Let $f, g : \{0, 1\}^n \rightarrow \{0, 1\}$ be bipermutive rules generating orthogonal Latin squares. Then, f and g are PWB

Lemma

Let $f, g : \{0, 1\}^n \rightarrow \{0, 1\}$ be bipermutive rules with generating functions $\varphi, \gamma : \{0, 1\}^{n-2} \rightarrow \{0, 1\}$. If φ and γ are PWB, then f and g are PWB as well

- ▶ **Remark:** φ, γ PWB: sufficient but not necessary condition for f, g to be PWB!
- ▶ Counterexamples already available for $n = 4$

Enumeration of PWB Generating Functions

- ▶ PWB generating functions of size $n-2 \Leftrightarrow$ balanced **quaternary** strings of size 2^{n-2}
- ▶ Example: $n = 5$, $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$

$$\Omega(\varphi) = (0, 1, 0, 1, 1, 0, 1, 0)$$

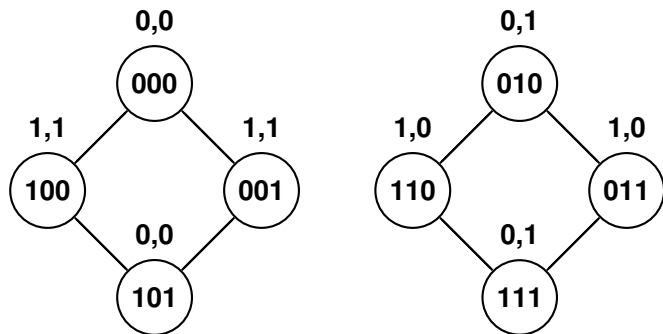
$$\Omega(\gamma) = (0, 1, 1, 0, 1, 0, 0, 1)$$

$$S_{\varphi,\gamma} = (1, 4, 3, 2, 4, 1, 2, 3)$$

- ▶ Each number from 1 to 4 appears $2^{5-4} = 2$ times
- ▶ The number of balanced quaternary strings of length 2^{n-2} is

$$\#\text{Bal}\mathcal{G}_n = \binom{2^{n-2}}{2^{n-4}} \cdot \binom{3 \cdot 2^{n-4}}{2^{n-4}} \cdot \binom{2^{n-3}}{2^{n-4}}$$

Enumeration of PWB Bipermutive Functions



- ▶ Bipermutivity: each c.c. has either $(0,0)/(1,1)$ or $(1,0)/(0,1)$ labels, oriented north-south or east-west
- ▶ PWB: number of $(0,0)/(1,1)$ and $(1,0)/(0,1)$ c.c. are equal
- ▶ Number of PWB pairs of bipermutive rules of size n :

$$\#\text{Bal}\mathcal{B}_n = \binom{2^{n-2}}{2^{n-3}} \cdot 2^{2^{n-2}}$$

Search Spaces Sizes

n	$\#\mathcal{B}_n$	$\#Bal\mathcal{G}_n$	$\#Bal\mathcal{B}_n$
3	16	0	8
4	256	24	96
5	65536	2520	17920
6	4294967296	63006300	843448320
7	$\approx 1.84 \cdot 10^{19}$	$\approx 9.96 \cdot 10^{15}$	$\approx 2.58 \cdot 10^{18}$

- ▶ Our results do not still allow to exhaustively search beyond $n = 6$, even by focusing on $Bal\mathcal{B}_n$
- ▶ We used a 40-core machine to span $Bal\mathcal{B}_n$, which took 22 hours to complete

Classification Results

n	LS_size	#total	#linear	#nonlinear	$(NI(f), NI(g), \#pairs)$
3	4×4	1	1	0	—
4	8×8	9	5	4	(4, 4, 4)
5	16×16	213	21	192	(4, 4, 96), (8, 8, 96)
6	32×32	66685	85	66600	(4, 4, 512), (12, 12, 17992), (8, 8, 4020), (16, 16, 28388), (20, 20, 14384), (4, 12, 8), (8, 16, 160), (12, 20, 128), (16, 24, 88)



Summing up:

- ▶ We considered the problem of exhaustively enumerating pairs of bipermutive CA generating orthogonal Latin squares, and classify them wrt nonlinearity
- ▶ We proved that pairwise balancedness is a necessary condition for two rules to generate OLS
- ▶ We used this condition to enumerate pairs up to size $n = 6$

Future directions:

- ▶ Find **sufficient** conditions for two rules to generate OLS
- ▶ Combinatorial encoding to evolve pairs of PWB bipermutive rules through **Genetic Algorithms** (preliminary results available in [Mariot17])

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