Building Correlation Immune Functions from Sets of MOCA

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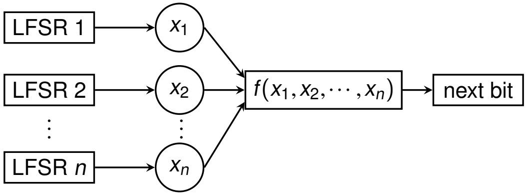
Correlation Immune Boolean Functions in Crypto

- **Boolean function**: mapping $f : \{0,1\}^n \rightarrow \{0,1\}$
- **Correlation Immunity of order t**: output of f is statistically independent from any subset of at most t inputs

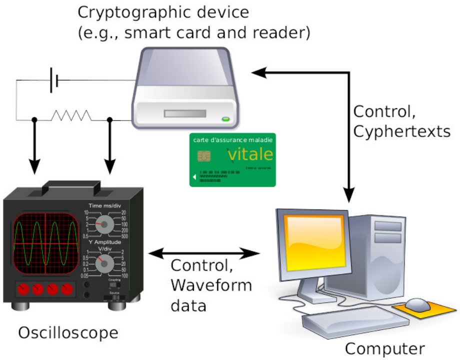


Applications in crypto:

- Combine the output of n LFSR for **stream encryption** [C21]
- CA-based **pseudorandom number generators** [L13, F14]

Countermeasures for Side-Channel Analysis

- **Idea:** attack a cipher by exploiting physical leakages in its implementation



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- CI functions are used for **masking** countermeasures in SCA [C12, K14]

Boolean Functions - Basic Representations

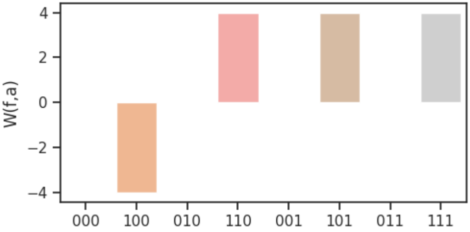
- Truth table: a 2^n-bit vector Ω_f specifying $f(x)$ for all $x \in \{0,1\}^n$

(x_1, x_2, x_3)	000	100	010	110	001	101	011	111
Ω_f	0	1	1	0	1	0	1	0

- Algebraic Normal Form (ANF): Sum (XOR) of products (AND)

$$f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3 \oplus x_2 x_3$$

- Walsh Transform: correlation with linear functions $a \cdot x$,
  $W(f,a) = \sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus a \cdot x}$ for all $a \in \{0,1\}^n$



Correlation Immunity: Walsh Characterization

- f is t-correlation immune iff $W_f(a) = 0$ for all a s.t. $1 \leq HW(a) \leq t$, where HW is the Hamming weight of a [X88]

Example: $t = 2$, $f(x) = x_1 \oplus x_2 \oplus x_3$

(x_1, x_2, x_3)	000	**100**	**010**	**110**	**001**	**101**	**011**	111
Ω_f	0	1	1	0	1	0	0	1
$\hat{F}(\omega)$	0	0	0	0	0	0	0	8

$$\Downarrow$$

f is 2-order correlation immune

- Relevance in side-channel: t-order CI functions $\Rightarrow$ Boolean masking resistant to SCA attacks of order t

Orthogonal Arrays (OA)

- (N,k,s,t) **Orthogonal Array**: $N \times k$ matrix A such that each t-uple occurs $\lambda = N/s^t$ times in each $N \times t$ submatrix.

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

1	0	0
0	0	0
0	1	0
0	0	1
0	1	1
1	1	1
1	0	1
1	1	0

Example: OA $(8,4,2,3)$

$\Rightarrow$ Each 3-bit vector $(x_1,x_2,x_3) \in \{0,1\}^3$ appears once in the submatrix with columns 1, 3, 4

- Applications in statistics, coding theory, cryptography

Correlation Immunity: OA Characterization

- **Support** of f: sets of input vectors x that map to 1 under f

Truth table

x_1	x_2	x_3	$f(x)$
0	0	0	0
0	0	1	**1**
0	1	0	**1**
0	1	1	0
1	0	0	**1**
1	0	1	0
1	1	0	0
1	1	1	**1**

Support

x_1	x_2	x_3
0	0	1
0	1	0
1	0	0
1	1	1

$$\Downarrow$$

$$OA(4,3,2,2)$$

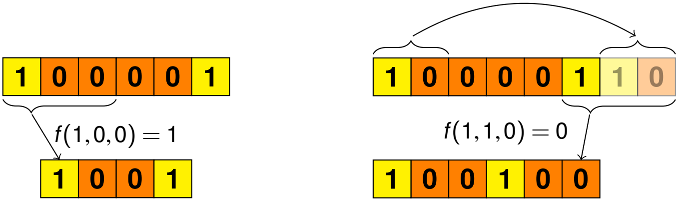
Theorem ([C92])

$f:\{0,1\}^n \rightarrow \{0,1\}$ is t-order CI $\Leftrightarrow$ Support of f is an $OA(N,n,2,t)$, with $N = |Supp(f)|$

Cellular Automata

- One-dimensional Cellular Automaton (CA): a discrete parallel computation model composed of a finite array of n cells

Example: $n = 6$, $d = 3$, $f(s_i, s_{i+1}, s_{i+2}) = s_i \oplus s_{i+1} \oplus s_{i+2}$ (rule 150)



No Boundary CA – NBCA

Periodic Boundary CA – PBCA

- Each cell updates its state $s \in \{0,1\}$ by evaluating a local rule $f : \{0,1\}^d \rightarrow \{0,1\}$ on itself and the $d-1$ cells on its right

Mutually Orthogonal Latin Squares (MOLS)

Definition

A *Latin square* is a $n \times n$ matrix where all rows and columns are permutations of $[n] = \{1, \cdots, n\}$. Two Latin squares are *orthogonal* if their superposition yields all the pairs $(x, y) \in [n] \times [n]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

1 1	3 4	4 2	2 3
4 3	2 2	1 4	3 1
2 4	4 1	3 3	1 2
3 2	1 3	2 1	4 4

- k-**MOLS**: set of k pairwise orthogonal Latin squares
- k-MOLS are equivalent to $OA(n^2, k, n, 2)$

Latin Squares through Bipermutive CA (1/2)

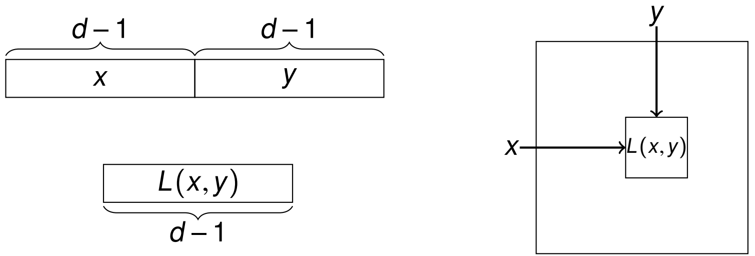
- Bipermutive CA: denoting $\mathbb{F}_2 = \{0,1\}$, local rule f is defined as

$$f(x_1, \cdots, x_d) = x_1 \oplus \varphi(x_2, \cdots, x_{d-1}) \oplus x_d$$

- $\varphi : \mathbb{F}_2^{d-2} \to \mathbb{F}_2$: generating function of f

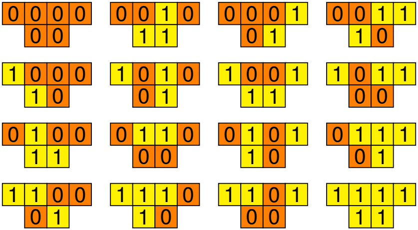
Lemma ([E93, M16])

A CA $F : \mathbb{F}_2^{2(d-1)} \to \mathbb{F}_2^d$ with bipermutive rule $f : \mathbb{F}_2^d \to \mathbb{F}_2$ generates a Latin square of order $N = 2^{d-1}$



Latin Squares through Bipermutive CA (2/2)

- Example: CA $F : \mathbb{F}_2^4 \to \mathbb{F}_2^2$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

k-**Mutually Orthogonal Cellular Automata** (MOCA): k bipermutive CA F, G generating a set of k-MOLS

Example with Linear CA: Rules 90-150

- Bipermutive Linear rule: $f(x) = x_1 \oplus a_2 x_2 \oplus \cdots \oplus a_{d-1} x_{d-1} \oplus x_d$
- Polynomial rule: $P_f(X) = 1 + a_2 X + \cdots + a_{d-1} X^{d-2} + X^{d-1}$

Theorem ([M20])

Two bipermutive linear rules generates OCA if and only if their associated polynomials are coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1 1	4 2	3 3	2 4
2 2	3 1	4 4	1 3
4 3	1 4	2 1	3 2
3 4	2 3	1 2	4 1

(c) Superposition

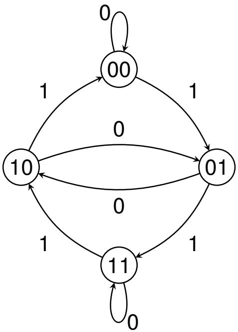
Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

De Bruijn graphs and bipermutative labelings

Definition

A labeling $l : E \to S$ for the de Bruijn graph $G_{m,n} = (V, E)$ over the set S is *bipermutative* if, for any vertex $v \in V$, the labels on the ingoing and outgoing edges of v form a permutation of S.

Example: $S = \{0, 1\}$, $m = n = 2$, $l_1((v_1, v_2), (u_1, u_2)) = v_1 \oplus u_2$



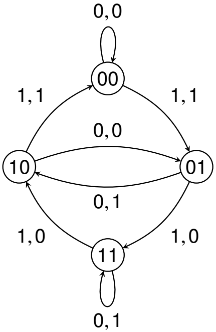
$(v_1, v_2) \to (u_1, u_2)$	l
$00 \to 00$	0
$10 \to 00$	1
$01 \to 10$	0
$11 \to 10$	1
$00 \to 01$	1
$10 \to 01$	0
$01 \to 11$	1
$11 \to 11$	0

OCA as Orthogonal Labelings

Definition

Two bipermutative labelings l_1, l_2 are *orthogonal* for $G_{m,n}$ over S if, for each pair $(x,y) \in S^n \times S^n$, there is *exactly one* path in $G_{m,n}$ of length n labelled by (x,y) under the superposed labeling $l_1.l_2$.

Example: $S = \{0,1\}$, $m = n = 2$, $l_{90} = v_1 \oplus u_2$, $l_{150} = v_1 \oplus u_1 \oplus u_2$



$(v_1,v_2) \to (u_1,u_2)$	l_1	l_2
$00 \to 00$	0	0
$10 \to 00$	1	1
$01 \to 10$	0	1
$11 \to 10$	1	0
$00 \to 01$	1	1
$10 \to 01$	0	0
$01 \to 11$	1	0
$11 \to 11$	0	1

Construction of CI functions from MOCA

Procedure:

1. **Input:** k**-MOCA** $F_1, \cdots, F_k : \mathbb{F}_2^{2b} \to \mathbb{F}_2^b$ **of diameter** $d = b + 1$
2. **Construct the set of** k**-MOLS over** $[2^b]$ **using the algorithms from [M17, M18]**
3. **"Linearize" the** k**-MOLS in a** $2^b \times k$ **OA** A
4. **Convert each entry in the OA in binary**
5. **Output: the binary array**

Rule 150

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

Rule 90

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$A =$

L_{90}	L_{150}
1	1
2	4
3	3
4	2
2	2
1	3
4	4
3	1
3	4
4	1
1	2
2	3
4	3
3	2
2	1
1	4

$\Longrightarrow$

L_{90}		L_{150}	
0	0	0	0
1	1	1	1
0	1	0	1
1	0	1	0
1	0	1	0
0	1	0	1
1	1	1	1
0	0	0	0
0	1	1	1
1	0	0	0
0	0	1	0
1	1	0	1
1	1	0	1
0	0	1	0
1	0	0	0
0	1	1	1

Lemma

The output array is an $OA(2^b, kb, 2, 2)$.

Computational Search Results

- **Consequence**: k-MOCA generate supports of Boolean functions with $n = kb$ variables with CI order at least 2
- Exhaustive search of 3-MOCA with $d = 4, 5$, $d = b + 1$
- Checked CI order with Walsh Transform

Table: Classification of correlation immune functions generated by 3-MOCA of diameter $d \in \{4, 5\}$.

d	#3-MOCA	n	w_H	CI	$\#CI$	Minw_H
4	2	9	64	3	2	20
5	36	12	256	3	27	24
5	36	12	256	4	9	24

- **Main finding**: all functions are at least 3-CI

Conclusions and Future Work

Wrapping up:

- We proved that k-MOCA generate correlation immune functions of order at least 2
- Experimentally, we noticed that k-MOCA functions actually have order at least 3

Future directions:

- Theoretically: are there MOCA that give CI functions with $t = 2$?
- Practically: reduce the Hamming weight of the functions, using evolutionary algorithms [M21]

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