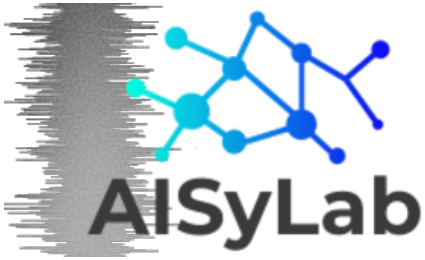

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Hip to Be (Latin) Square: Maximal Period Sequences from Orthogonal CA

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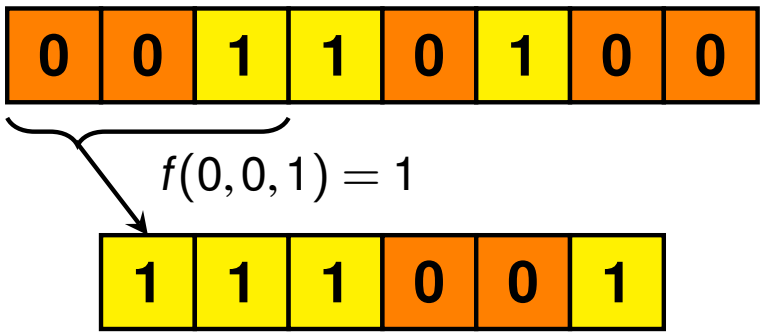
Cellular Automata

- One-dimensional Cellular Automaton (CA): a discrete parallel computation model composed of a finite array of n cells

Example: $n = 8$, $d = 3$, $f(s_i, s_{i+1}, s_{i+2}) = s_i \oplus s_{i+2}$ (rule 90)



No Boundary CA – NBCA

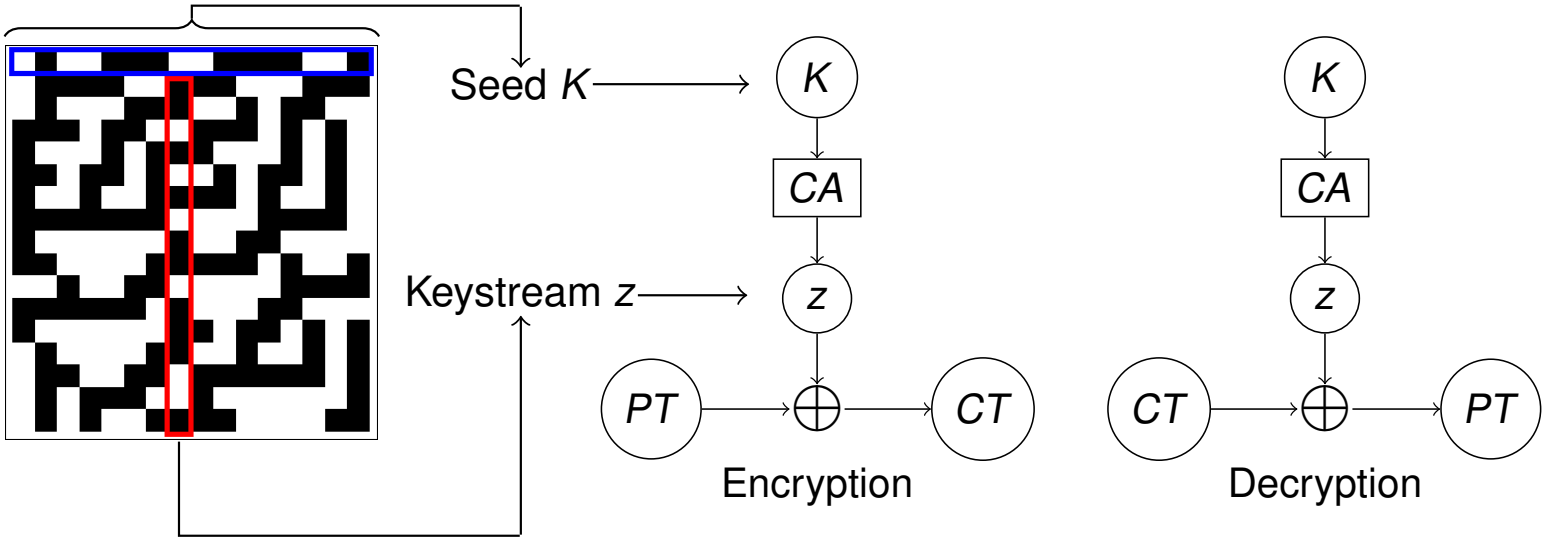
x_i, x_{i+1}, x_{i+2}	$f(x_i, x_{i+1}, x_{i+2})$
000	0
100	1
010	0
110	1
001	1
101	0
011	1
111	0

Truth table – Rule 90

- Each cell updates its state $s \in \{0,1\}$ by applying a local rule $f : \{0,1\}^d \to \{0,1\}$ to itself and the $d-1$ cells on its right

CA-based Crypto History: Wolfram's PRNG

- CA-based Pseudorandom Generator (PRG) [W86]: central cell of rule 30 CA used as a stream cipher keystream



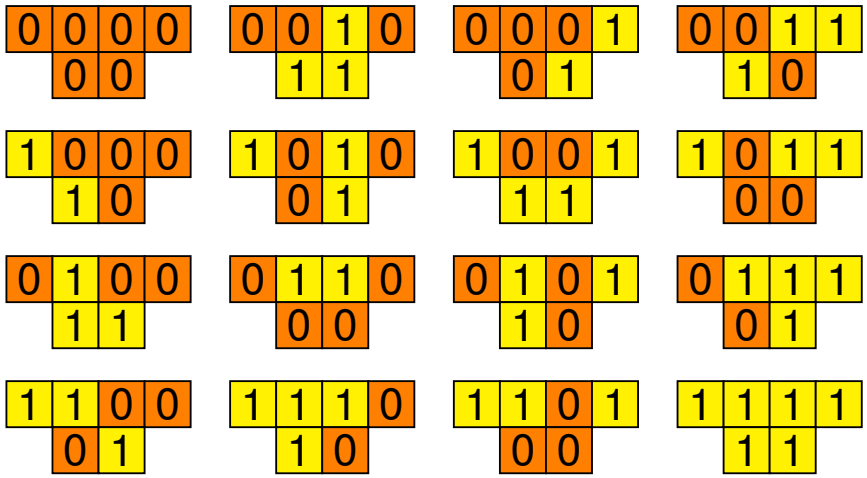
- This CA-based PRNG was later shown to be vulnerable [MS91]
- More recent works [LM13, FIMY14, LM4] tried to fix it using larger rules with better crypto properties

Latin Squares by Bipermutive CA [E93, MFL16]

- Bipermutive CA: local rule f defined as

$$f(x_1, \cdots, x_d) = x_1 \oplus \varphi(x_2, \cdots, x_{d-1}) \oplus x_d$$

- Example: CA $F : \mathbb{F}_2^4 \to \mathbb{F}_2^2$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

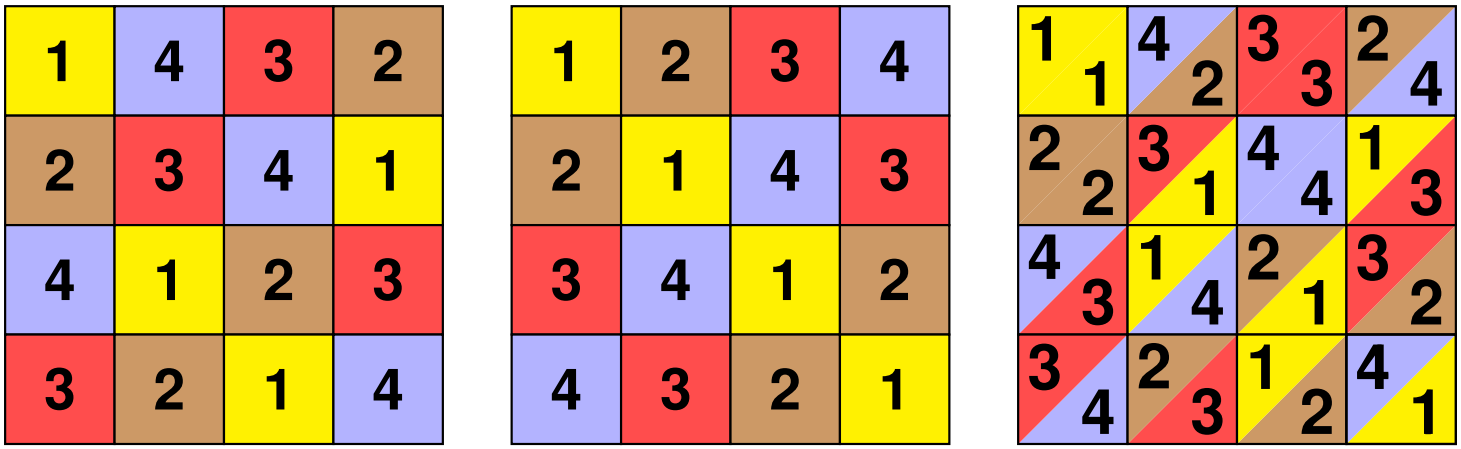
Orthogonal Cellular Automata (OCA): pair of bipermutive CA generating two orthogonal Latin squares

OCA by Linear CA

- Bipermutive Linear rule: $f(x) = x_1 \oplus a_2 x_2 \oplus \cdots \oplus a_{d-1} x_{d-1} \oplus x_d$
- Polynomial rule: $P_f(X) = 1 + a_2 X + \cdots + a_{d-1} X^{d-2} + X^{d-1}$

Theorem ([MGFL20])

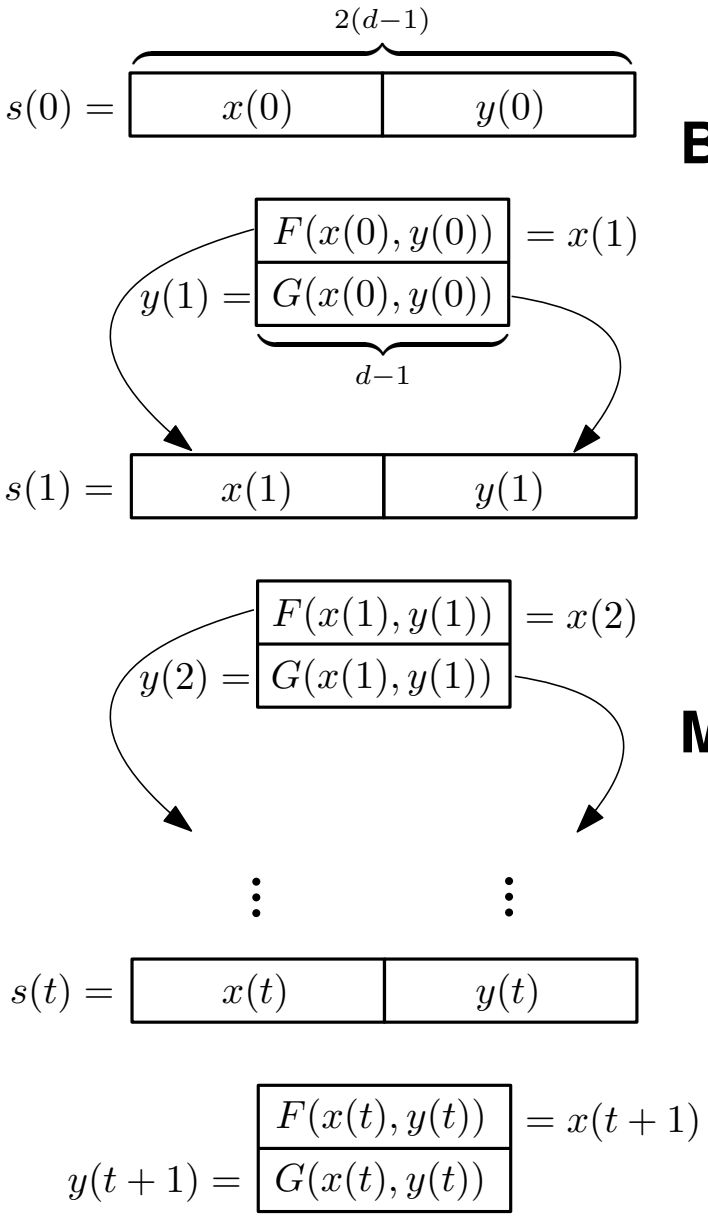
Two linear bipermutive CA F, G are OCA if and only if their associated polynomials $P_f(X), P_g(X)$ are relatively prime.



(a) Rule 150 (b) Rule 90 (c) Superposition

Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Pseudorandom Generator based on OCA



Basic Idea:

- Start from random $(x(0), y(0))$ and evaluate two OCA F, G over it
- Use the outputs $F(x(0), y(0))$ and $G(x(0), y(0))$ as new OCA inputs
- Continue to iterate the system

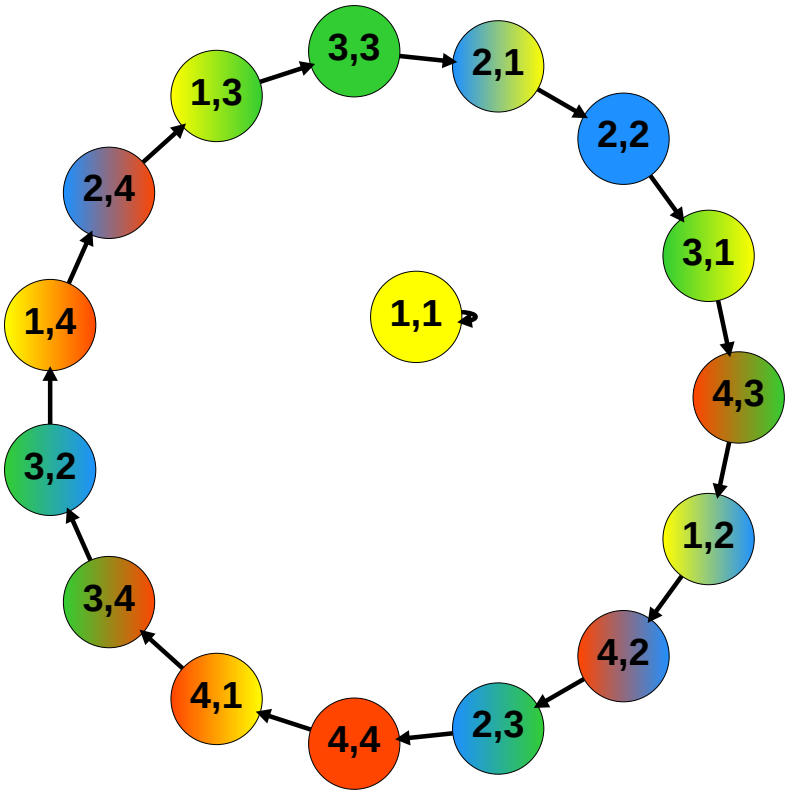
Motivation:

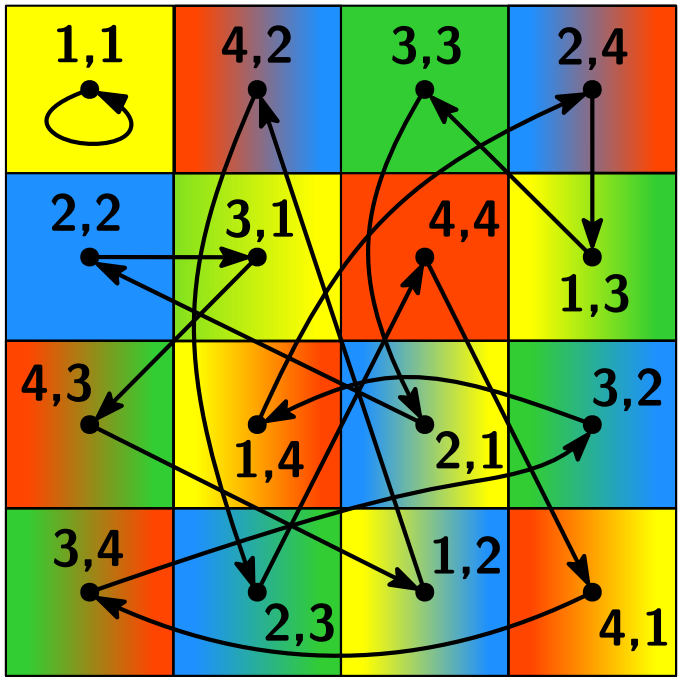
- The system is always reversible (because of orthogonality)
- Orthogonality ensure a minimum degree of *diffusion*

Longest Periods in OCA

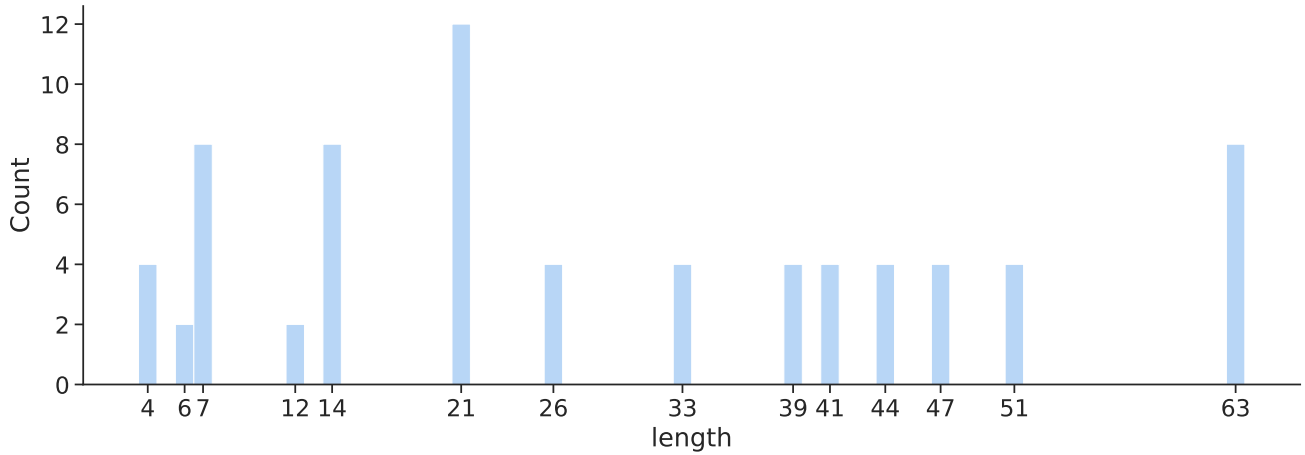
Research Question: How do we choose F and G to get a *maximum period length* of $2^{2(d-1)}$?

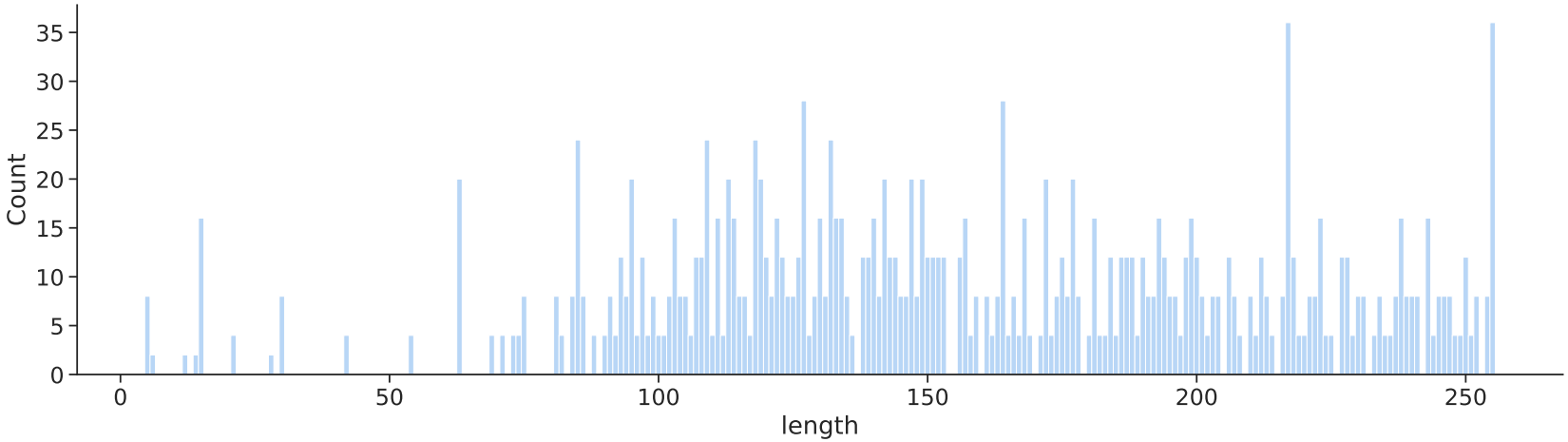
Example: $d = 3$, rules 90 and 150





Distribution of Maximum Periods for $d = 4, 5$





Main Remark: Best upper bound reached is $2^{2(d-1)} - 1$

The Case of Linear OCA

- For linear OCA F, G, finding an upper bound boils down to determine the order of the *Sylvester Matrix*:

$$M_{F,G} = \begin{pmatrix} a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_1 & \cdots & a_d \\ b_1 & \cdots & b_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_1 & \cdots & b_d & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_1 & \cdots & b_d \end{pmatrix}.$$

- We devised a *combinatorial algorithm* to efficiently enumerate all such matrices of maximum order

Results – Linear OCA Enumeration

Table: Number of maximal period linear OCA pairs of diameter $d \leq 11$.

d	n	$2^{2n}-1$	$\#\mathcal{LOCA}_d$	$\#m\mathcal{LOCA}_d$	Time
2	1	3	0	–	–
3	2	15	1	1	$<$ 1s
4	3	63	5	1	$<$ 1s
5	4	255	21	3	$<$ 1s
6	5	1023	85	15	$<$ 1s
7	6	4095	341	42	3.967s
8	7	16383	1365	181	59.162s
9	8	65535	5461	572	18m59.302s
10	9	262143	21845	1872	5h56m10.208s
11	10	1048575	87381	5899	4d16h27m22.126s

Conclusions

Recap of main findings:

- Orthogonal CA seems to represent an interesting way to generate pseudorandom sequences with long periods
- The longest periods seem to occur in the case of linear OCA
- Upper bounding the periods of linear OCA is equivalent to finding the order of a Sylvester matrix

Open problems:

- Study the number of maximum order Sylvester matrices (new sequence added in the OEIS [O21])
- Characterize which pairs of polynomials induce maximum order Sylvester matrices
- Study the periods of nonlinear OCA, possibly using an evolutionary approach [MPJL17, MPJL18]
- Generalize to CA-based Latin *hypercubes* [GM20]

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