Artificial Intelligence and Security Lab
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Enumerative combinatorics problems for cryptographic primitives based on CA

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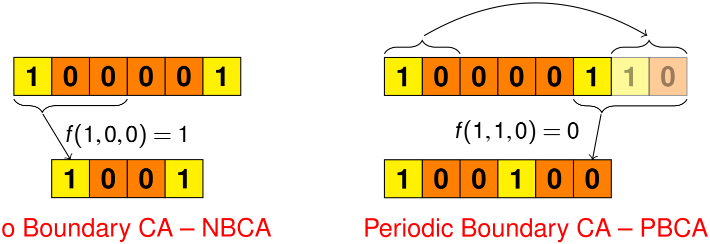
Cellular Automata

- One-dimensional Cellular Automaton (CA): a discrete parallel computation model composed of a finite array of n cells

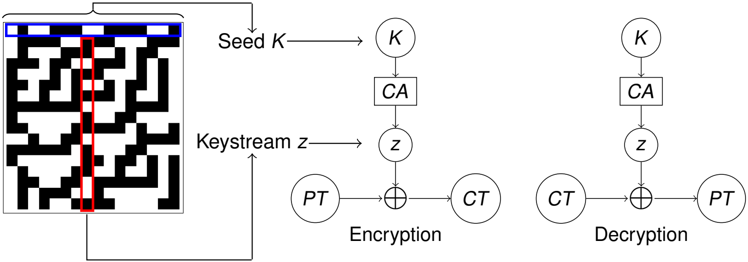
Example: $n = 6$, $d = 3$, $\omega = 0$, $f(s_i, s_{i+1}, s_{i+2}) = s_i \oplus s_{i+1} \oplus s_{i+2}$ (rule 150)



- Each cell updates its state $s \in \{0,1\}$ by applying a local rule $f : \{0,1\}^d \to \{0,1\}$ to itself, the ω cells on its left and the $d - 1 - \omega$ cells on its right

CA-based Crypto History: Wolfram's PRNG

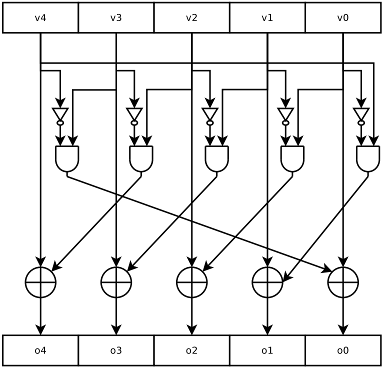
- CA-based Pseudorandom Generator (PRG) [W86]: central cell of rule 30 CA used as a stream cipher keystream



- This CA-based PRNG was later shown to be vulnerable [MS91]

CA-Based Crypto History: KECCAK χ S-box

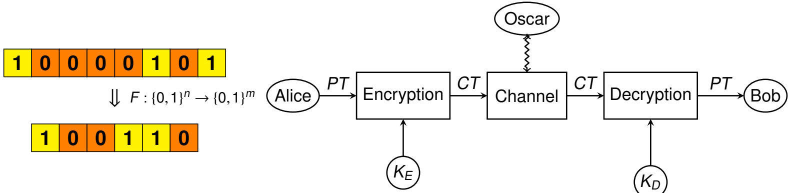
- Local rule: $\chi(x_1, x_2, x_3) = x_1 \oplus (1 \oplus (x_2 \cdot x_3))$ (rule 210)
- Invertible for every odd size n of the CA [DGV94]



- Used as a PBCA with $n = 5$ in the KECCAK specification of SHA-3 standard [BDPV11]

Motivations

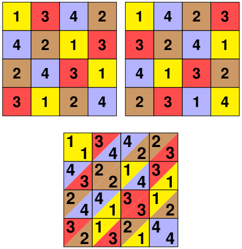
General Research Goal: Investigate cryptographic primitives defined by Cellular Automata



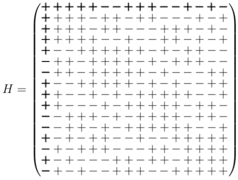
Why CA, anyway?

1. **Security from Complexity**: CA can yield very complex dynamical behaviors, depending on the local rule
2. **Efficient implementation**: Leverage CA parallelism and locality for lightweight cryptography

CA-based Primitives Considered



(a) Orthogonal Arrays & Orthogonal Latin Squares

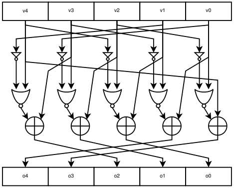


$$\Omega_f = (0,0,0,0,0,1,1,0,0,0,1,1,0,1,0,1)$$

$$\Downarrow$$

$$f(x_1,x_2,x_3,x_4) = x_1x_3 \oplus x_2x_3 \oplus x_2x_4$$

(b) Hadamard matrices & Bent functions



(c) S-boxes & Vectorial Boolean functions

- For each primitive, we will survey the main results and point out open problems of enumerative combinatorics

Part 1: Orthogonal Arrays & Orthogonal Latin Squares

Orthogonal Arrays (OA)

- (N,k,s,t) **Orthogonal Array**: $N \times k$ matrix over s symbols s.y. each t-uple occurs $\lambda = N/s^t$ times in each $N \times t$ submatrix

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

1	0	0
0	0	0
0	1	0
0	0	1
0	1	1
1	1	1
1	0	1
1	1	0

Example: OA $(8,4,2,3)$

$\Rightarrow$ Each 3-bit vector $(x_1, x_2, x_3) \in \{0,1\}^3$ appears once in the submatrix with columns 1, 3, 4

- **Crypto Applications**: threshold secret sharing schemes, masking for side-channel attacks

Orthogonal Latin Squares (OLS)

Definition

A *Latin square* is a $n \times n$ matrix where all rows and columns are permutations of $[n] = \{1, \cdots, n\}$. Two Latin squares are *orthogonal* if their superposition yields all the pairs $(x, y) \in [n] \times [n]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

1 1	3 4	4 2	2 3
4 3	2 2	1 4	3 1
2 4	4 1	3 3	1 2
3 2	1 3	2 1	4 4

- k pairwise OLS are denoted as k-MOLS (**Mutually Orthogonal Latin Squares**)
- k-MOLS are **equivalent** $OA(n^2, k, n, 2)$

Latin Squares through Bipermutive CA (1/2)

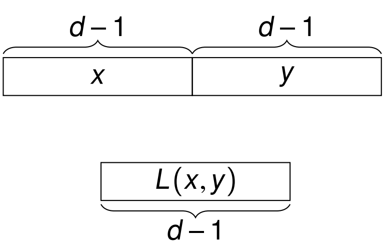
- Bipermutive CA: denoting $\mathbb{F}_2 = \{0,1\}$, local rule f is defined as

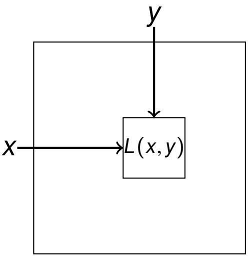
$$f(x_1, \cdots, x_d) = x_1 \oplus \varphi(x_2, \cdots, x_{d-1}) \oplus x_d$$

- $\varphi : \mathbb{F}_2^{d-2} \to \mathbb{F}_2$: generating function of f

Lemma ([MGFL20])

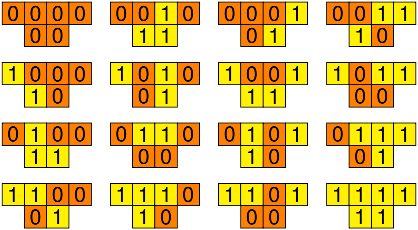
A CA $F : \mathbb{F}_2^{2(d-1)} \to \mathbb{F}_2^d$ with bipermutive rule $f : \mathbb{F}_2^d \to \mathbb{F}_2$ generates a Latin square of order $N = 2^{d-1}$





Latin Squares through Bipermutive CA (2/2)

- Example: CA $F : \mathbb{F}_2^4 \to \mathbb{F}_2^2$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

Mutually Orthogonal Cellular Automata (MOCA): set of k bipermutive CA generating k-MOLS

MOCA by Linear CA

- Bipermutive Linear rule: $f(x) = x_1 \oplus a_2 x_2 \oplus \cdots \oplus a_{d-1} x_{d-1} \oplus x_d$
- Polynomial rule: $P_f(X) = 1 + a_2 X + \cdots + a_{d-1} X^{d-2} + X^{d-1}$

Theorem ([MGFL20])

A set of k bipermutive linear CA are k-MOCA if and only if their associated polynomials are pairwise coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1, 1	4, 2	3, 3	2, 4
2, 2	3, 1	4, 4	1, 3
4, 3	1, 4	2, 1	3, 2
3, 4	2, 3	1, 2	4, 1

(c) Superposition

Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Counting linear CA-based (M)OLS

- Number of CA-based OLS pairs of diameter $d = n+1$ over $\mathbb{F}_q$:

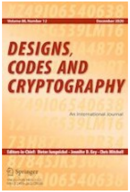
$$a_n = q(q-1)^3 \frac{q^{2n-2}-1}{q^2-1} + (q-1)(q-2)$$

- For $\mathbb{F}_2$, a_n coincides with OEIS sequence A002450
- Size of the biggest MOCA family of diameter $d = n+1$:

$$N_{n,q} = I_{n,q} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} I_{k,q}, \text{ where } I_{n,q} = \frac{1}{n} \sum_{e|n} \mu(e) \cdot q^{\frac{n}{e}}$$

- Results published in [MGFL20]:

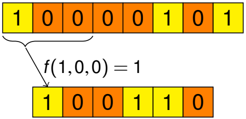
L. Mariot, M. Gadouleau, E. Formenti, and A. Leporati. Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)



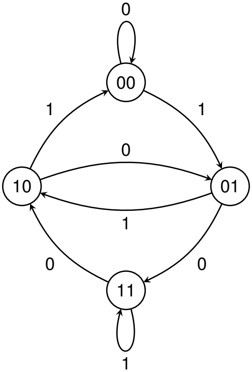
Generalization to nonlinear CA

- The previous results depend on the *linearity* of the local rules
- **Open Problem**: count and enumerate OLS and MOCA families generated by *nonlinear* bipermutive rules
- **Direction**: investigate the paths on the *de Bruijn graph*

Example: $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)



- CA input vector ⇔ path on the (overlapped) *vertices*
- CA output vector ⇔ path on the edges

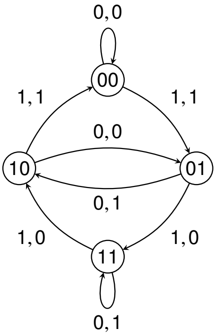


Orthogonal labelings

Definition

Two bipermutative labelings l_1, l_2 are *orthogonal* for $G_{m,n}$ over S if, for each pair $(x,y) \in S^n \times S^n$, there is *exactly one* path in $G_{m,n}$ of length n labelled by (x,y) under the superposed labeling $l_1.l_2$.

Example: $S = \{0,1\}$, $m = n = 2$, $l_1 = v_1 \oplus u_2$, $l_2 = v_1 \oplus u_1 \oplus u_2$



$(v_1,v_2) \to (u_1,u_2)$	l_1	l_2
$00 \to 00$	0	0
$10 \to 00$	1	1
$01 \to 10$	0	1
$11 \to 10$	1	0
$00 \to 01$	1	1
$10 \to 01$	0	0
$01 \to 11$	1	0
$11 \to 11$	0	1

Open Problems

Problem (Counting)

Given $m, n \in \mathbb{N}$, what is the number $N(m,n)$ of orthogonal pairs of bipermutative labelings for $G_{m,n}$?

Problem (Enumeration)

Find an algorithm that enumerates only $N(m,n)$ of orthogonal pairs of bipermutative labelings for $G_{m,n}$.

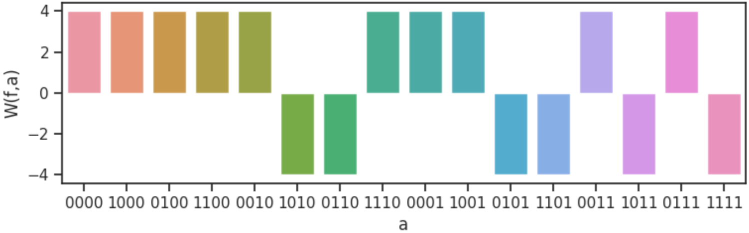
Part 2: Hadamard matrices & Bent functions

Hadamard Matrices & Bent functions

- Hadamard Matrix: a $n \times n$ matrix with ± 1 entries s.t. $H \cdot H^\top = I_n$. Example for order $n = 4$:

$$H = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}$$

- Walsh Transform of $f : \mathbb{F}_2^n \to \mathbb{F}_2$: $W(f,a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus a \cdot x}$
- Bent function: "flattest" Walsh spectrum, $W_f(a) = \pm 2^{\frac{n}{2}}$



Example: $f(x_1, x_2, x_3, x_4) = x_1 x_3 + x_1 x_4 + x_2 x_4$

Constructions of Bent Functions

- **Relevance in crypto**: bent functions reach the highest possible *nonlinearity*
- Number of Boolean functions of n variables: 2^{2^n}
- $\Rightarrow$ too huge for exhaustive search when $n > 5$!

In practice, one can resort to *algebraic constructions*

- *Primary constructions*: new functions are built from scratch (e.g., Maiorana-McFarland construction)
- *Secondary constructions*: new functions are obtained from existing ones (e.g., Rothaus's construction)

Hadamard Matrices characterizing Bent Functions

Theorem (Dillon, 1974)

Given $f : \{0,1\}^n \to \{0,1\}$ and $\hat{f}(x) = (-1)^{f(x)}$. Define the $2^n \times 2^n$ matrix H for all $x, y \in \{0,1\}^n$ as:

$$H(x,y) = \hat{f}(x \oplus y)$$

Then, f is a bent function if and only if H is a Hadamard matrix.

Example: $f(x_1, x_2) = x_1 x_2$

x_1	x_2	$x_1 x_2$
0	0	0
1	0	0
0	1	0
1	1	1

$$H = \begin{pmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{pmatrix}$$

Hadamard Matrices from MOLS

Theorem (Bush, 1973)

Given a set of t MOLS of order $N = 2t$, and A the associated $OA(t,2t)$, define the $4t^2 \times 4t^2$ matrix H as follows:

$$H(i,j) = \begin{cases} +1, & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and } \exists k \in \{1,\cdots,t\} \text{ s.t. the column} \\ & k \text{ of } A \text{ has the same symbol in rows } i \text{ and } j \\ +1, & \text{otherwise} \end{cases}$$

for $i,j \in \{1,\cdots,4t^2\}$. Then, H is a symmetric Hadamard matrix.

- **Remark:** the Hadamard matrix constructed from a MOLS family in general *does not* correspond to a bent function

From Linear CA to Bent Functions

- **Question**: Are the MOLS arising from linear CA suitable for constructing bent functions?
- We consider only CA over $\mathbb{F}_q$ with $q = 2^l$, $l \in \mathbb{N}$
- The order of the Hadamard matrix must be $4t^2 = 2^n$
- We need t coprime polynomials of degree b:

$$2^{lb} = 2t \Leftrightarrow lb = 1 + \log_2 t$$

- Since both l and b are integers, $t = 2^w$ for $w \in \mathbb{N}$

From Linear CA to Bent Functions

Theorem

Let H be the Hadamard matrix of order $2^{2(w+1)}$ defined by the t LBCA $F_1, \cdots F_t : \mathbb{F}_q^{2b} \to \mathbb{F}_q^b$, and define $f : \mathbb{F}_2^n \to \mathbb{F}_2$, $n = 2(w+1)$ as:

$$f(x) = \begin{cases} 0 \ , & \textit{if } x = 0 \\ 1 \ , & \textit{if } x \neq 0 \textit{ and } \exists k \in \{1, \cdots, t\} \textit{ s.t. } F_k(x) = 0 \\ 0 \ , & \textit{otherwise} \end{cases} ,$$

Then, it holds that:

$$H(x,y) = \hat{f}(x \oplus y)$$

and thus f is a bent function

Remark: The linearity of the CA is crucial to grant this result

Example

$A =$

L_1	L_2
1	1
2	4
3	3
4	2
2	2
1	3
4	4
3	1
3	4
4	1
1	2
2	3
4	3
3	2
2	1
1	4

$$H = \begin{pmatrix}
+&+&+&+&+&-&-&+&+&+&-&-&+&-&+&- \\
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\end{pmatrix}$$

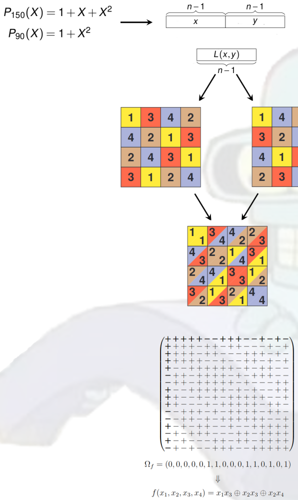
$$\Omega_f = (0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1)$$

$$\Downarrow$$

$$f(x_1, x_2, x_3, x_4) = x_1x_3 \oplus x_2x_3 \oplus x_2x_4$$

Figure 3: Example of bent function of $n = 4$ variables generated by the $t = 2$ MOLS of order $2t = 4$ defined by the LBCA with rule 90 and 150, respectively. The two Latin squares are represented on the left in the OA form. The first row and the first column of the Hadamard matrix H coincide with the polarity truth table of the function.

Existence and Counting



Combinatorial questions addressed in [GMP20]:

- **Existence:** for even n, a large enough family of coprime polynomials exists iff the degree is either 1 or 2
- **Counting:** how many families of this kind exist (= number of CA-based bent functions)

Currently submitted to Designs, Codes and Cryptography

Open Problems

Other Remarkable findings [GMP20]:

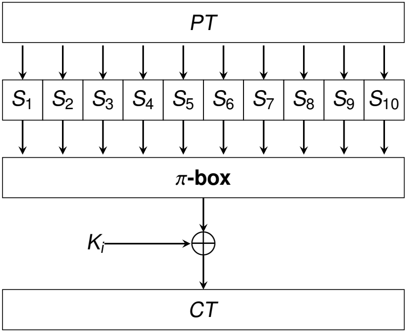
- Bent functions from this construction belong to the *Partial Spread class* $\mathcal{PS}^-$
- For degree 1, the resulting class of bent functions coincides with the *Desarguesian spread*

Open problems:

- Investigate the case of degree 2, to see if our functions are equivalent to other known classes
- Is our construction generalizable to *nonlinear* CA?

Part 3: S-Boxes & Vectorial Boolean functions

Block Ciphers: Substitution-Permutation Network



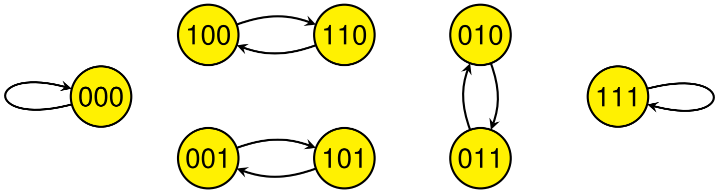
- $S_i : \mathbb{F}_2^n \to \mathbb{F}_2^n$ are S-boxes, or vectorial Boolean functions, providing confusion
- The S-boxes must:
  - be bijective
  - have high nonlinearity
  - have low differential uniformity

- **Research line**: use periodic CA to design S-boxes
- In [MPLD19], *Genetic Programming* is used to design CA-based S-boxes, with $n = d$ and $4 \leq n \leq 8$

Reversible CA

- An (infinite) CA is *reversible* (RCA) if its global rule $F : \{0,1\}^n \to \{0,1\}^n$ is *bijective* and F^{-1} is also a CA [H69]
- For finite PBCA, there exist *globally invertible* rules that give RCA only for certain lengths n

Example: $n = 3$, $d = 3$, $\omega = 0$, $f(x_i, x_{i+1}, x_{i+2}) = x_i \oplus x_{i+1} \cdot x_{i+2} \oplus x_{i+2}$



- Local rules resulting in RCA for every size n of the array are also called *locally invertible* [DGV94]

Marker CA

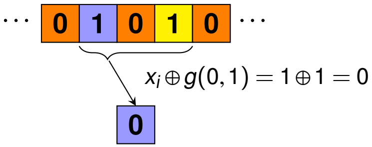
- The local rule f of marker CA is defined as follows:

$$f(x_{i-\omega}\cdots x_{i-1}x_i x_{i+1}\cdots x_{i-\omega+d-1}) = x_i \oplus g(x_{i-\omega}\cdots x_{i-1}x_{i+1}\cdots x_{i-\omega+d-1})$$

- Equivalently: the *support* of g defines the *markers* for which the central cell *flips* its state

Example: $d = 3$, $\omega = 0$, $f(x_i, x_{i+1}, x_{i+2}) = x_i \oplus x_{i+1} \cdot x_{i+2} \oplus x_{i+2}$

x_{i+1}	x_{i+2}	$g(x_{i+1}, x_{i+2})$
0	0	0
1	0	0
0	**1**	**1**
1	1	0

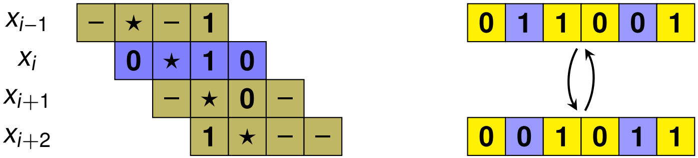


Marker: $01 \Rightarrow \star 01$ **Flipping landscape**

Conserved Landscape Marker CA

- *Conserved Landscape*: each cell in a flipping landscape must be in the *same* landscape after applying the CA global rule

Example: $d = 4$, $\omega = 1$, Landscape: $0 \star 10$

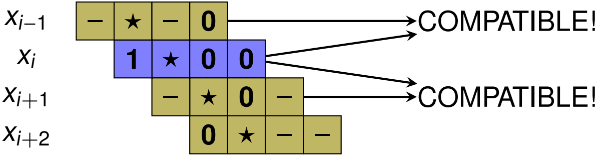


Landscape tabulation | Example of orbit of period 2

- A landscape is conserved if it is *incompatible* with all its *neighborhood landscapes*

Optimizing Conserved Landscape Rules

- **Idea**: Use *Evolutionary Algorithms* to investigate the class of conserved landscape CA [MPJL20]
- **First Objective:** *minimize* the number of neighborhood landscapes that are compatible with each flipping landscape

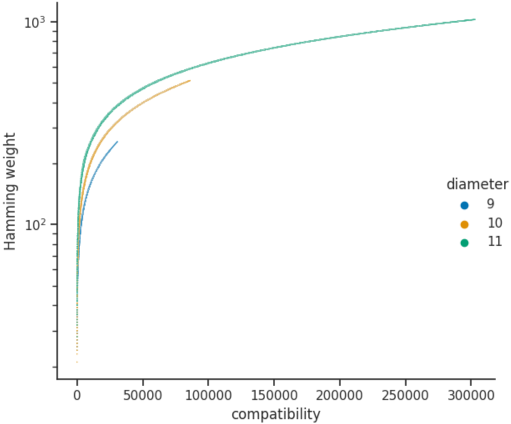


- **Second objective:** *maximize* the Hamming weight of the generating function (related to S-box nonlinearity)

$$g(x) = \boxed{0\,|\,0\,|\,1\,|\,0\,|\,1\,|\,0\,|\,0\,|\,0} \Rightarrow \text{weight: } 2$$

Experimental Findings

- **Main finding**: *The more a marker CA rule is reversible, the lower its Hamming weight must be*

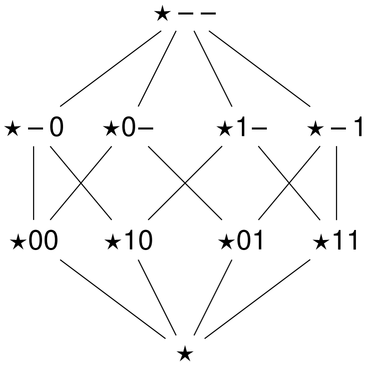


- **Open problem 1**: determine an *upper bound* on the Hamming weight for conserved landscape rules

Open Problems

- **Open problem 2**: count the number of such rules
- Compatible landscapes induce a *partial order* $\leq_C$:

$$L \leq_C M \Leftrightarrow l_i = m_i \text{ or } l_i \in \{0,1\} \text{ and } m_i = -, \quad 0 \leq i \leq d-1$$



- A conserved landscape rule is an *antichain* on this poset
- Counting antichains in general finite posets is $\#\mathcal{P}$-complete!
- Is there an efficient way to count them on this poset?

Wrap-up & Conclusions

Conclusions

We surveyed three CA-based cryptographic primitives:

- Orthogonal Arrays and Mutually Orthogonal Latin Squares,
- Hadamard Matrices and Bent Functions,
- S-boxes defined by reversible CA,

showing several open combinatorial problems related to them

Several **other directions** exist on this research line, such as:

- *Latin Hypercubes* defined by (linear) CA [GM20]
- *Asynchrony Immunity* as a relevant cryptographic property for CA-based S-boxes [MMD20]

Thank you!

References

[BDPV11] Bertoni, G., Daemen, J., Peeters, M., Van Assche, G.: The Keccak reference. `http://keccak.noekeon.org/` (2011)

[DGV94] Daemen, J., Govaerts, R., Vandewalle, J.: An efficient nonlinear shift-invariant transformation. In Proceedings of the 15th Symposium on Information Theory in the Benelux, pp. 108-115 (1994)

[GM20] Gadouleau, M., Mariot, L.: Latin Hypercubes and Cellular Automata. Proceedings of Automata 2020, pp. 139-151 (2020)

[GMP20] Gadouleau, M., Mariot, L., Picek, S.: Bent Functions from Cellular Automata. IACR Cryptol. ePrint Arch. 2020: 1272 (2020)

[H69] Hedlund, G.A.: Endomorphisms and Automorphisms of the Shift Dynamical Systems. Mathematical Systems Theory 3(4): 320–375 (1969)

[MPLD19] Mariot, L. Picek, S., Leporati, A., Jakobovic, D.: Cellular Automata Based S-Boxes. *Cryptography and Communications* 11(1): 41-62 (2019)

[MGFL20] Mariot, L., Gadouleau, M. Formenti, E., Leporati A.: Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)

[MMD20] Mariot, L., Manzoni, L., Dennunzio, A.: Search space reduction of asynchrony immune cellular automata. Nat. Comput. 19(2): 287-293 (2020)

[MPJL20] Mariot, L., Picek, S., Jakobovic, D., Leporati A.: An Evolutionary View on Reversible Shift-Invariant Transformations. Proceedings of EuroGP 2020, pp. 118-134 (2020)

[MS91] Meier, W., Staffelbach, O.: Analysis of Pseudo Random Sequence Generated by Cellular Automata. In EUROCRYPT, Vol. 91, pp. 186-200 (1991)

[W86] Wolfram, S.: Random Sequence Generation by Cellular Automata. Adv. Appl. Math. 7(2), 123–169 (1986)