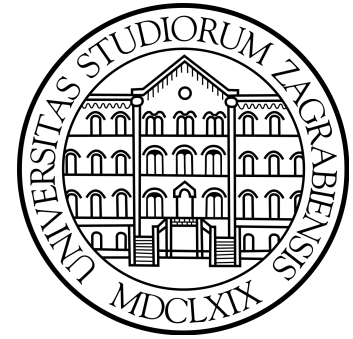

Evolutionary Algorithms for the Design of Orthogonal Latin Squares based on CA

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Latin Squares

Definition

A *Latin square* of order N is a $N \times N$ matrix L such that in every row and in every column each number of $[N] = \{1, \cdots, N\}$ occurs exactly once

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Orthogonal Latin Squares (OLS)

Definition

Two Latin squares L_1 and L_2 of order N are *orthogonal* if their superposition yields all the pairs $(x, y) \in [N] \times [N]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	4	1

1,1	3,4	4,2	2,3
4,3	2,2	1,4	3,1
2,4	4,1	3,3	1,2
3,2	1,3	2,1	4,4

Applications of sets of Mutually orthogonal Latin squares (MOLS):

- Cryptography (secret sharing schemes, authentication codes)
- Coding theory (MDS codes)

Design of OLS

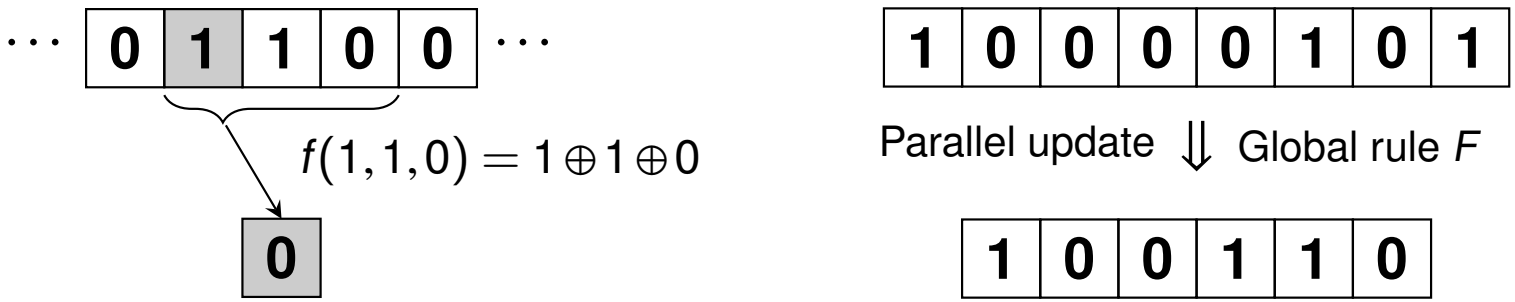
- Existing constructions of OLS mostly based on algebraic methods [Keedwell15]
- No work addressing the design of OLS via evolutionary techniques
- Possible reasons and challenges:
  - **Representation issues**: how to encode a pair of Latin squares?
  - **Variation operators**: how to cross two individuals and still get a pair of Latin squares?
  - **Search space analysis**: the number of Latin squares/OLS is not even known for generic N

A different perspective: Cellular Automata (CA)

Definition

One-dimensional CA: triple $\langle m, n, f \rangle$ where $m \in \mathbb{N}$ is the number of cells on a one-dimensional array, $n \in \mathbb{N}$ is the neighborhood and $f : \{0,1\}^n \to \{0,1\}$ is the local rule.

Example: $m = 8$, $n = 3$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)



Notation: $\Omega(f)$ is the 2^n-bit string representing the truth table of f

$$f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3 \Rightarrow \Omega(f) = 01101001$$

Latin Squares through Bipermutive CA (1/2)

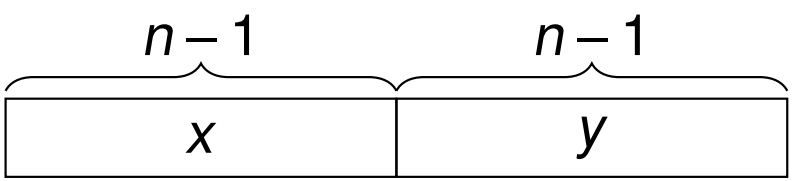
- Bipermutive CA: local rule f is defined as

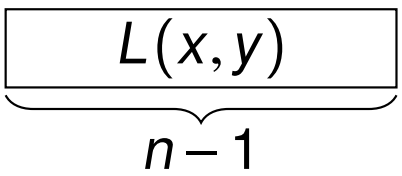
$$f(x_1, \cdots, x_n) = x_1 \oplus \varphi(x_2, \cdots, x_{n-1}) \oplus x_n$$

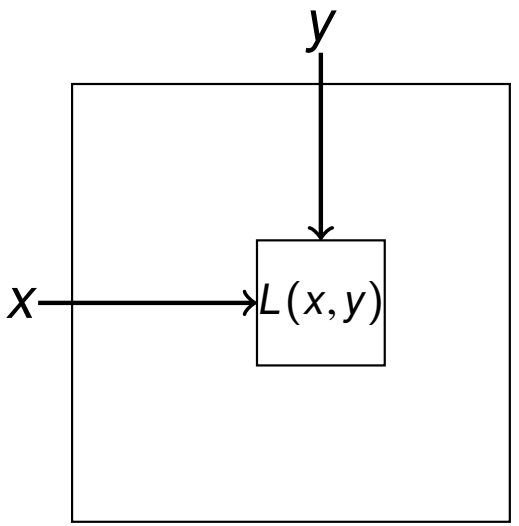
- $\varphi : \{0,1\}^{n-2} \rightarrow \{0,1\}$: generating function of f

Lemma ([Eloranta93, Mariot16])

Let $\langle 2(n-1), n, f \rangle$ be a CA with bipermutive rule. Then, the global rule F generates a Latin square of order $N = 2^{n-1}$

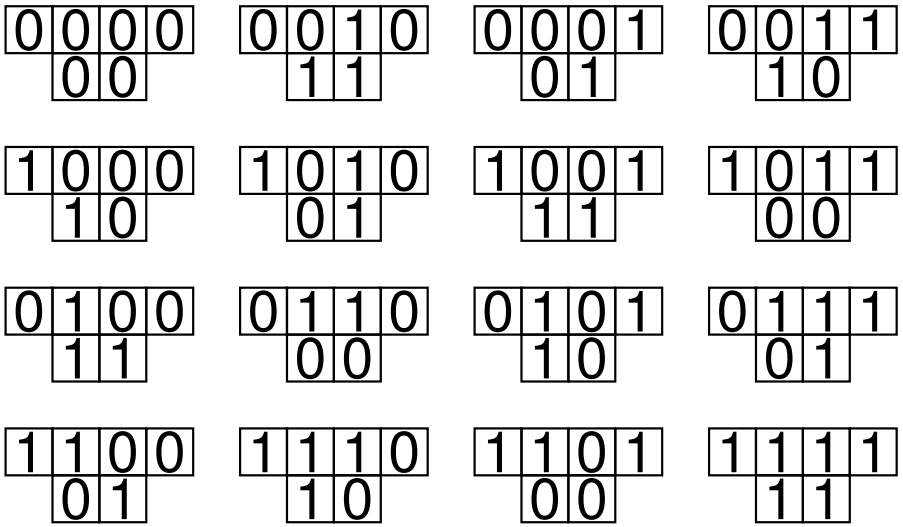






Latin Squares through Bipermutive CA (2/2)

- Example: CA $\langle 4, 3, f \rangle$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

Motivations and Goals

- Construction of OLS solved for linear CA [Mariot16]
- MOLS arising from nonlinear constructions have relevance in cheater-immune Secret Sharing Schemes [Tompa88]

Goal: Design OLS based on CA by evolving pairs of nonlinear bipermutive local rules through GA and GP

Twofold motivation:

- **Theoretical:** Understand the mathematical structure of the space of nonlinear CA-based OLS
- **EC perspective:** Source of new problems for evolutionary algorithms

Nonlinearity of CA Local Rules (Boolean Functions)

- Affine function: $l(x_1, \cdots, x_n) = a \oplus a_1 x_1 \oplus \cdots \oplus a_n x_n$, $a, a_i \in \{0,1\}$
- Nonlinearity of f: Hamming distance of $\Omega(f)$ from the set of all affine functions
- Walsh transform of f: given $\omega \in \{0,1\}^n$,

$$W_f(\omega) = \sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus \omega \cdot x} \text{ , where } \omega \cdot x = \bigoplus_{i=1}^{n} \omega_i \cdot x_i$$

Definition

Let $f : \{0,1\}^n \to \{0,1\}$. The *nonlinearity* of f is defined as

$$Nl(f) = 2^{n-1} - \frac{1}{2} \max_{\omega \in \{0,1\}^n} \{|W_f(\omega)|\}$$

Search Space Size

- Number of Boolean functions of n variables: $\mathcal{F}_n = 2^{2^n}$
- Bipermutive rules of size $n \Leftrightarrow$ Generating functions of size $n-2$ (which are $\mathcal{F}_{n-2} = 2^{2^{n-2}}$)
- Pairs of bipermutive rules of size n: $\mathcal{B}_n = 2^{2^{n-1}} = \mathcal{F}_{n-1}$

n	3	4	5	6	7	8
$\mathcal{B}_n$	16	256	65536	$\approx 4.3 \times 10^9$	$\approx 1.8 \cdot 10^{19}$	$\approx 3.4 \cdot 10^{38}$
$N \times N$	4×4	8×8	16×16	32×32	64×64	128×128
$\#OLS$	8	72	1704	533480	?	?

Remark: Exhaustive enumeration possible up to $n = 6$

Fitness Functions (1/2)

- $\#rep(L_1, L_2)$: Number of occurrences of each pair (except the first one) in the superposition of Latin squares L_1 and L_2

1	3	4	2
4	2	1	3
2	4	3	1
2	3	4	1

(a) L_1

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) L_2

4,1	1,4	2,3	3,2
3,2	2,3	1,4	4,1
1,4	4,1	3,2	2,3
2,3	3,2	4,1	1,4

(c) $\#rep(L_1, L_2) = 12$

- Let φ, γ be the generating functions of two bipermutive CA, and let L_φ, L_γ be the associated Latin squares

First fitness function: minimize $fit_1(\varphi, \gamma) = \#rep(L_\varphi, L_\gamma)$

Fitness Functions (2/2)

- **Remark**: fit_1 does not consider the nonlinearity of φ and γ!
- Nonlinearity penalty factor:

$$NlPen(\varphi,\gamma) = \begin{cases} 0 \ , & \text{if } Nl(\varphi) > 0 \ \ AND \ \ Nl(\gamma) > 0 \\ 1 \ , & \text{if } Nl(\varphi) = 0 \ \ XOR \ \ Nl(\gamma) = 0 \\ 2 \ , & \text{if } Nl(\varphi) = 0 \ \ AND \ \ Nl(\gamma) = 0 \end{cases}$$

Second fitness function: minimize

$$fit_2(\varphi,\gamma) = \#rep(L_\varphi, L_\gamma) + NlPen(\varphi,\gamma) \cdot N^2$$

- The N^2 scaling factor balances the range of $\#rep(L_\varphi, L_\gamma)$, which is $\{0, \cdots, N^2\}$

GA Encoding: Single Bitstring

- Let $\varphi, \gamma : \{0,1\}^{n-2} \rightarrow \{0,1\}$ be a pair of generating functions, with 2^{n-2}-bit truth tables $\Omega(\varphi), \Omega(\gamma)$, and let $||$ denote concatenation

$$\textbf{First GA encoding}: enc_1(\varphi, \gamma) = \Omega(\varphi)||\Omega(\gamma)$$

- Hence, each chromosome is the concatenation of length 2^{n-1} of the truth tables of the generating functions

Example:

$$\varphi(x_1, x_2, x_3) = x_1 \oplus x_3 \Rightarrow \Omega(\varphi) = (0,1,0,1,1,0,1,0)$$
$$\gamma(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3 \Rightarrow \Omega(g) = (0,1,1,0,1,0,0,1)$$
$$enc_1(\varphi, \gamma) = (0,1,0,1,1,0,1,0,0,1,1,0,1,0,0,1)$$

- Classic GA variation operators like one-point crossover and bit-flip mutation are applied in this case

GA & GP Encodings: Double Bitstring/Double Tree

- Idea: Keep the generating functions separated and evolve them independently

Second GA encoding: $enc_2(\varphi, \gamma) = (\Omega(\varphi), \Omega(\gamma))$

- We use the same idea for GP: the genotype is composed of two trees $T(\varphi)$ and $T(\gamma)$ representing φ and γ

GP encoding: $enc_{GP}(\varphi, \gamma) = (T(\varphi), T(\gamma))$

- Classic GA and GP variations operators are applied independently on each of the two components

GA Encoding: Balanced Quaternary Strings (1/2)

Definition

$f, g : \{0,1\}^n \to \{0,1\}$ are pairwise balanced (PWB) if

$$\left|(f,g)^{-1}(0,0)\right| = \left|(f,g)^{-1}(1,0)\right| =$$
$$= \left|(f,g)^{-1}(0,1)\right| = \left|(f,g)^{-1}(1,1)\right| = 2^{n-2}$$

Example:

- $f(x_1, x_2, x_3) = x_1 \oplus x_3$ (Rule 90)
- $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)

$$\Omega(f) = (0,1,0,1,1,0,1,0) \ ,$$
$$\Omega(g) = (0,1,1,0,1,0,0,1) \ .$$

Each of the pairs $(0,0),(1,0),(0,1),(1,1)$ occurs $2^{3-2} = 2$ times

GA Encoding: Balanced Quaternary Strings (2/2)

- Experimental observations on exhaustive search:
  - Two bipermutive CA generate OLS $\Rightarrow$ the local rules are PWB
  - Generating functions are PWB $\Rightarrow$ the local rules are PWB

Third GA encoding: $enc_3(\varphi, \gamma)$ is a quaternary string of length 2^{n-2} where each number from 1 to 4 occurs 2^{n-4} times

Example: $n = 5, (0,0) \mapsto 1, (1,0) \mapsto 2, (0,1) \mapsto 3, (1,1) \mapsto 4$

$$\Omega(\varphi) = (0,1,0,1,1,0,1,0)$$
$$\Omega(\gamma) = (0,1,1,0,1,0,0,1)$$
$$enc_3(\varphi, \gamma) = (1,4,3,2,4,1,2,3)$$

- Balancedness-preserving variation operators for GA:
  - Crossover: use counters to keep track of the multiplicities of the 4 values in the offspring (as in [Mariot15])
  - Mutation: use a swap-based operator

Experimental settings

Common Parameters:

- Problem instances: rules of $n = 7$ and $n = 8$ variables
- Termination condition: 300000 fitness evaluations
- Each experiment is repeated over 50 independent runs
- Selection operator: steady-state with 3-tournament operator

GA Parameters:

- Population size: 30 individuals
- Crossover and mutation probabilities: $p_c = 0.95$, $p_m = 0.2$

GP Parameters:

- Boolean operators: AND, OR, XOR, XNOR, NOT, IF
- Population size: 500 individuals
- Mutation probability: $p_m = 0.5$

Results

- (GA, n, enc_i): GA experiment with CA rules of n variables and encoding enc_i, fitness function fit_1
- (GP, n, fit_i): GP experiment with CA rules of n variables and encoding enc_{GP}, fitness function fit_i

Exp.	avg fit	std fit	#opt	#lin	#nlin
$(GA, 7, enc_1)$	520.32	360.16	12/50	0	12
$(GA, 7, enc_2)$	565.44	389.03	15/50	0	15
$(GA, 7, enc_3)$	392.64	328.47	18/50	0	18
$(GA, 8, enc_1)$	4165.44	604	1/50	0	1
$(GA, 8, enc_2)$	4222.16	125.03	0/50	0	0
$(GA, 8, enc_3)$	4696.48	135.51	0/50	0	0
$(GP, 7, fit_1)$	0	0	50/50	50	0
$(GP, 7, fit_2)$	0	0	50/50	0	50
$(GP, 8, fit_1)$	0	0	50/50	47	3
$(GP, 8, fit_2)$	0	0	50/50	0	50

Discussion

For GP:

- GP always manages to converge to an optimal solution
- ... but under fit_1, all solutions found are linear!
- Possible explanation: GP first converges to linear pairs (since it has the XOR operator), then OLS are easily found

On the other hand, for GA:

- GA converged just once for $n = 8$ and the performances for $n = 7$ are worse than GP
- ... but all solutions found are nonlinear, even under fit_1

Conclusions and Future Directions

Wrapping up:

- We addressed for the first time the problem of designing orthogonal Latin squares based on nonlinear CA through GA and GP
- We experimented with three encodings for the candidate solutions and two fitness functions
- The problem seems to be easy for GP up to $n = 8$ variables, less so for GA (who converges only up to $n = 7$ variables)

Future directions:

- Investigate the behaviour of GP under fit_1 (why does it always find linear solutions?)
- Design other encodings for the candidate solutions of GA

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