Counting Coprime Polynomials over Finite Fields with Formal Languages and Compositions of Natural Numbers

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Coprime Polynomials

Object: pairs of binary polynomials of degree $n \in \mathbb{N}$:

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \ ,$$
$$g(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + x^n \ ,$$

where $a_i, b_i \in GF(2) = \mathbb{F}_2 = \{0, 1\}$

$$f, g \in \mathbb{F}_2[x] \text{ are } \textbf{coprime} \Leftrightarrow \gcd(f, g) = 1$$

Applications cryptography and coding theory:

- *Discrete logarithms* in finite fields [C84]
- Decoding *alternant codes* [F95]

Euclid’s Algorithm

Check if $\gcd(f, g) = 1 \Rightarrow$ **Euclid’s algorithm**

Example: $n = 3, \quad f(x) = x^3 + x^2 + x + 1, \quad g(x) = x^3 + 1$

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$$f(x) = q(x) \cdot g(x) + r(x)$$

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$x^3 + x^2 + x + 1 = 1 \cdot (x^3 + 1) + (x^2 + x)$

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$$x^2 + x = x \cdot (x + 1) + 0$$

Compact notation:

$$(x^3 + x^2 + x + 1, x^3 + 1) \xrightarrow{1} (x^3 + 1, x^2 + x) \xrightarrow{x+1} (x^2 + x, x + 1) \xrightarrow{x} (x + 1, 0)$$

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Example: $n=3, \quad f(x)=x^3+x^2+x+1, \quad g(x)=x^3+1$

$$f(x)=q(x)\cdot g(x)+r(x)$$

$$\begin{aligned}
&x^3+x^2+x+1=1\cdot(x^3+1)+(x^2+x)\\
&x^3+1=(x+1)\cdot(x^2+x)+(x+1)\\
&x^2+x=x\cdot(x+1)+0
\end{aligned}$$

Compact notation:

$(x^3+x^2+x+1,x^3+1)\xrightarrow{1}(x^3+1,x^2+x)\xrightarrow{x+1}(x^2+x,x+1)\xrightarrow{x}(x+1,0)$

$$gcd(f,g)=x+1 \Rightarrow (f,g)\ \textbf{not}\ \text{coprime}$$

DilcuE's Algorithm

- **Remark:** (f,g) can be recovered from $(x+1,0)$ with the same quotients in reverse order
- Called **DilcuE's algorithm** by Benjamin and Bennett [BB07]

$(x+1,0) \xrightarrow{x}$

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- Suppose we change the **last** remainder from 0 to 1:

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- By construction, (f',g') **are coprime**

Counting by Bijection

In essence: bijection for coprime/non-coprime pairs over $\mathbb{F}_2$:

1. Apply Euclid to (f, g)
2. If the last remainder is 0, change it to 1. Otherwise, set it to the second-last remainder
3. Apply DilcuE's algorithm to the reversed quotients

Theorem ([BB07, R00])

Let $f, g \in \mathbb{F}_2[x]$ of degree n be randomly chosen. Then, the probability that $\gcd(f, g) = 1$ is $\frac{1}{2}$.

In other words: the number of coprime pairs is 2^{2n-1}

Enter the complication

We require now that both f and g have a **nonzero constant term**:

$$f(x) = \mathbf{1} + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \ ,$$
$$g(x) = \mathbf{1} + b_1 x + \cdots + b_{n-1} x^{n-1} + x^n \ .$$

Problems:

1. *Count* all such pairs
2. *Enumeration algorithm*

Remark: the trick above does not work!

non-coprime $\leftrightarrow$ **coprime**

$$(x^3 + x^2 + x + \mathbf{1}, x^3 + \mathbf{1}) \leftrightarrow (x^3 + x + \mathbf{1}, x^3 + x^2 \ (+\mathbf{0}))$$

... Why do we want to do that?

n-MOLS by Cellular Automata

- Bipermutive Linear rule: $f(x) = x_1 \oplus a_1 x_2 \oplus \cdots \oplus a_{n-1} x_{n-1} \oplus x_n$
- Associated Polynomial: $P_f(X) = 1 + a_1 X + \cdots + a_{n-1} X^{n-1} + X^n$

Theorem ([MGFL20])

n bipermutive linear CA generates a set of n-MOLS if and only if their associated polynomials are pairwise coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(b) Rule 90

1, 1	4, 2	3, 3	2, 4
2, 2	3, 1	4, 4	1, 3
4, 3	1, 4	2, 1	3, 2
3, 4	2, 3	1, 2	4, 1

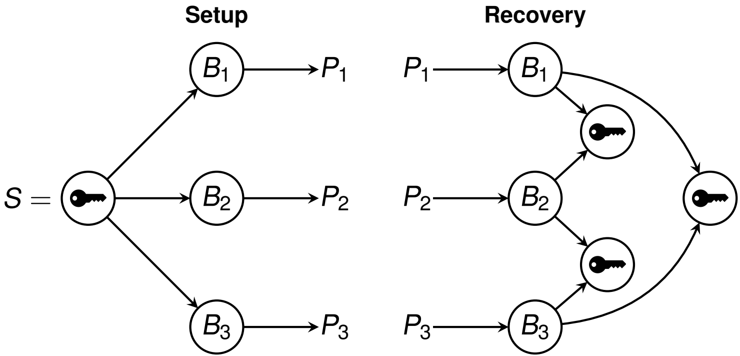
(c) Superposition

Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Secret Sharing Schemes (SSS)

- (k, n)–Secret sharing scheme share a secret S among n players, so that at least k of them are required to recover S
- $(2, n)$–schemes $\Leftrightarrow$ families of n-MOLS

Example: $(2,3)$–scheme



Asking for clues (7 years ago...)

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Dear Arthur, what do you think of this complication?

Luca

Asking for clues (7 years ago...)

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Arthur Benjamin

Dear Luca, off the top of my head, there are $q^2 - 1$ equivalence classes, all of which are co-prime except one? But I may be wrong.

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Dear Luca, off the top of my head, there are $q^2 - 1$ equivalence classes, all of which are co-prime except one? But I may be wrong.

?

One MONTH later...

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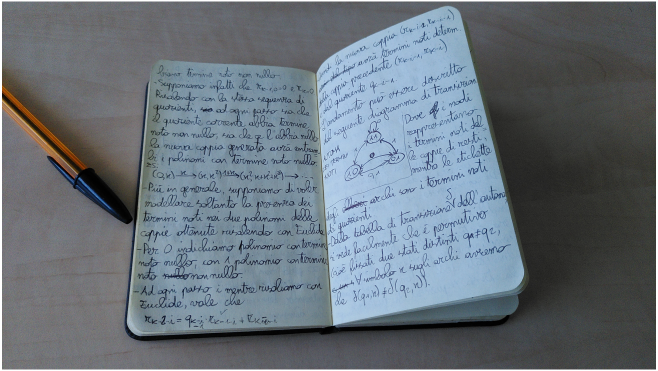
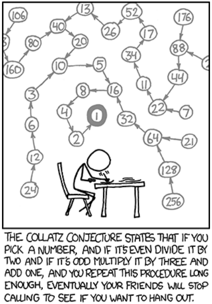
... He was indeed right! But took me several weeks to prove it



Sadly, the clue was not enough to solve the counting problem



S `https://xkcd.com/710/`

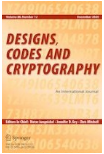
- Number of coprime polynomial pairs of degree n and nonzero constant term:

$$a(n) = 4^{n-1} + a(n-1) = \frac{4^{n-1}-1}{3}$$
$$= 0, 1, 5, 21, 85, \ldots$$

- Corresponds to OEIS A002450

- Generalized for any finite field $\mathbb{F}_q$ in [MGFL20] (but enumeration not addressed)

L. Mariot, M. Gadouleau, E. Formenti, and A. Leporati. Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)



Problem Structure

Strategy: characterize the *sequences* of quotients that gives only $(1,1)$ coprime pairs when starting from the remainders $(1,0)$

Three parts of the problem:

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Three parts of the problem:

$$
\begin{array}{llllll}
q_1 \rightarrow & \overbrace{x^{d_1}}^{\textit{degrees}} & + \overbrace{q_{1,d_1-1}x^{d_1-1} + \cdots + q_{1,1}x}^{\text{middle terms}} & + & \overbrace{s_1}^{\text{constant terms}} \\
q_2 \rightarrow & x^{d_2} & + q_{2,d_2-1}x^{d_2-1} + \cdots + q_{2,1}x & + & s_2 \\
\vdots \rightarrow & \vdots & + \quad \vdots \quad + \cdots + \quad \vdots & + & \vdots \\
q_k \rightarrow & x^{d_k} & + q_{k,d_k-1}x^{d_k-1} + \cdots + q_{k,1}x & + & s_k
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Problem Structure

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\end{array}
$$

Notation: $r_i, r_{i+1} \rightarrow$ consecutive remainders produced by Euclid's algorithm at step i. Step $i+1$:

$$r_i(x) = q_{i+1}(x)r_{i+1}(x) + r_{i+2}(x)$$

Finite State Automaton of Remainders

$$r_i(x) = q_{i+1}(x) r_{i+1}(x) + r_{i+2}(x)$$

- $(c_i, c_{i+1}) \to$ constant terms of r_i and r_{i+1}
- $s_{i+1} \to$ constant term of q_{i+1}
- $\delta((c_i, c_{i+1}), s_{i+1}) \to$ *next* pair (c_{i+1}, c_{i+2})

(c_i, c_{i+1})	s_{i+1}	$\delta((c_i, c_{i+1}), s_{i+1})$
$(1,1)$	0	$(1,1)$
$(1,1)$	1	$(1,0)$
$(1,0)$	0	$(0,1)$
$(1,0)$	1	$(0,1)$
$(0,1)$	0	$(1,0)$
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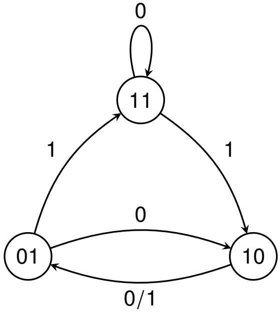
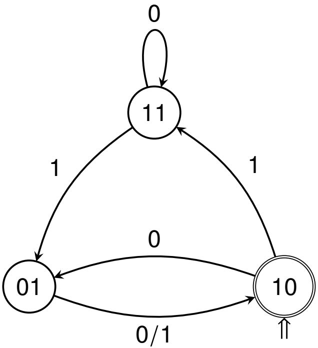
Remark: the pair $(0,0)$ *never* occurs



The Regular Language of Constant Terms Sequences



Inverse FSA

- The FSA is *permutative*: for DilcuE's, simply reverse the arrows
- **Initial state**: 10
- **Final state**: 11 (but we can use 10)

Regular Expression of the Language:

$$L = (0(0+1) + (10^*1(0+1)))^*$$

Enumeration/counting of Constant Terms Sequences

- **Enumeration**: generate all words of length k [M97]
- **Counting**: exploit *algebraic language theory*

Transform $L = (0(0+1)+(10^*1(0+1)))^*$ in a FPS as follows:

- $0,1 \Rightarrow X$
- $+,\cdot \Rightarrow +,\cdot$
- $^* \Rightarrow \frac{1}{1-X}$

Generating Function:

$$\sum_{k=0}^{\infty} a_k \cdot X^k = \frac{1-X}{1-X-2X^2} ,$$

Closed Form:

$$a_k = \frac{2^k + 2\cdot(-1)^k}{3}$$

Sequences of quotients' degrees

Second part: Characterize the *degrees* of the quotients

Example: $n = 4$, $\{1, x, x^2, x, 1\}$

$$(1,0) \xrightarrow{1} (1,1) \xrightarrow{x} (x+1,1) \xrightarrow{x^2} (x^3+x^2+1, x+1) \xrightarrow{x}$$
$$(x^4+x^3+1, x^3+x^2+1) \xrightarrow{1} (x^4+x^2+1, x^4+x^3+1)$$

Sum of degrees: $1+2+1 = 4$, $k = 3$

Question: what are the combinations of *ordered sums* of n?

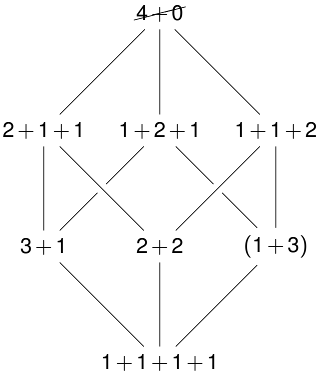
$\Rightarrow$ **compositions** of $n \in \mathbb{N}$

Quotients' degrees as compositions of n

- **Representation:** $n-1$ *boxes* that can be either "+" or ","

$$1\overbrace{\square 1 \square \ldots \square 1 \square}^{n-1}1$$

- **Example:** $1,1+1,1 \to 1+2+1 \quad (n=4,\, k=3)$



- We remove the top of the poset
- **Enumeration**: generate all binary strings of length n with k 1s
- **Counting**: $\binom{n-1}{k-1}$

Enumeration Algorithm

- **Third part**: middle terms are *free*
- once k is fixed, all three parts are *independent*

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So for **enumeration**, given $n \in \mathbb{N}$:

For each composition *comp* of n of length k (except $k = 0$) do:

- Generate all quotients' sequences of *comp* (2^{n-k})
- For each quotients' sequence *seq* do:
  - For each constant term sequence of length k do:
    - Add the constant terms to the quotients
    - Apply DilcuE's from $(1, 0)$ by applying *seq*

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$$\sum_{k=2}^{n} \underbrace{2^{n-k}}_{\text{middle}} \cdot \underbrace{\binom{n-1}{k-1}}_{\textit{degrees}} \cdot \underbrace{\frac{2^k + 2\cdot(-1)^k}{3}}_{\textit{constant}}$$

Conclusions and Future Work

Summing up:

- Enumeration more complicated with nonzero constant terms
- We divided the problem in three tasks:
  1. sequences of constant terms ($\Rightarrow$ regular language)
  2. sequences of degrees ($\Rightarrow$ compositions)
  3. sequences of middle terms ($\Rightarrow$ free)
- Results informally published in [FM22]

Future directions:

- Generalize to any finite field $\mathbb{F}_q$ and to *m*-tuples of polynomials
- Applications to cryptography [GM20, M21, GMP22]

Thank you!

References

[BB07] Benjamin, A.T., Bennett, C.D.: The probability of relatively prime polynomials. Mathematics Magazine 80(3): 196-202 (2007)

[C84] Coppersmith, D.: Fast evaluation of logarithms in fields of characteristic two. IEEE Trans. Inf. Theory 30(4): 587-593 (1984)

[F95] Fitzpatrick, P.: On the key equation. IEEE Trans. Inf. Theory 41(5): 1290-1302 (1995)

[FM22] Formenti, E., Mariot, L.: An Enumeration Algorithm for Binary Coprime Polynomials with Nonzero Constant Term. CoRR abs/2207.00406 (2022)

[GMP22] Gadouleau, M., Mariot, L., Picek, S.: Bent functions in the partial spread class generated by linear recurring sequences. Des. Codes Cryptogr. In press. DOI: `https://doi.org/10.1007/s10623-022-01097-1` (2022)

[GM20] Gadouleau, M., Mariot, L.: Latin Hypercubes and Cellular Automata. Proceedings of Automata 2020, pp. 139-151 (2020)

[GR11] Ghorpade, S. R., Ram, S.: Block companion Singer cycles, primitive recursive vector sequences, and coprime polynomial pairs over finite fields. Finite Fields Their Appl. 17(5): 461-472 (2011)

[M97] E. Mäkinen: On Lexicographic Enumeration of Regular and Context-Free Languages. Acta Cybern. 13(1): 55-61 (1997)

[M21] Mariot, L.: Hip to Be (Latin) Square: Maximal Period Sequences from Orthogonal Cellular Automata. In: Proceedings of CANDAR 2021, pp. 29-37 (2021)

[MGFL20] Mariot, L., Gadouleau, M. Formenti, E., Leporati A.: Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)

[R00] Reifegerste, A.: On an involution concerning pairs of polynomials in $\mathbb{F}_2$. J. Combin. Theory Ser. A 90, 216-220 (2000)