Behind the scenes of a new construction for bent functions

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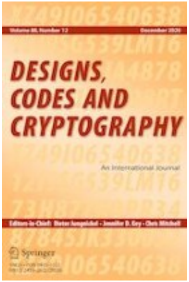
Summary

Introduction: an (unlucky) paper

M. Gadouleau, L. Mariot, S. Picek. Bent functions in the partial spread class generated by linear recurring sequences. Des. Codes and Cryptogr. (in press) DOI: 10.1007/s10623-022-01097-1



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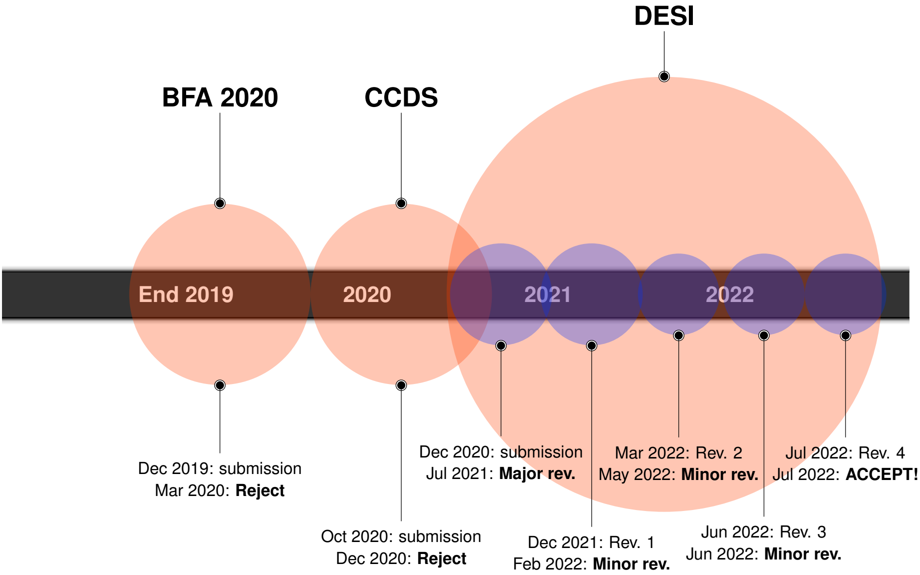
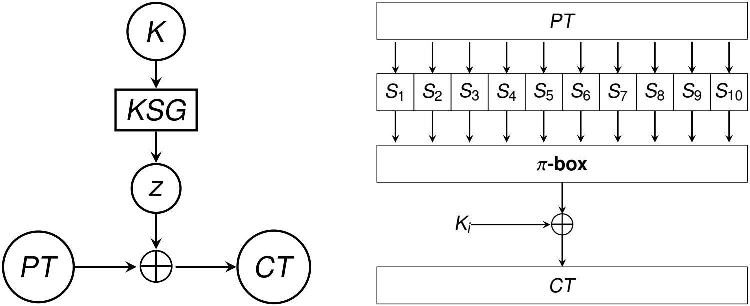
Peer-review Timeline



Boolean Functions in Symmetric Ciphers



(a) Stream cipher (b) Block cipher

Boolean functions $f : \{0,1\}^n \to \{0,1\}$ are used in [C21]

- **Stream ciphers**, to design the *keystream generator* (KSG)
- **Block ciphers**, as the coordinate functions of *S-boxes* (S_i)

Boolean Functions - Basic Representations

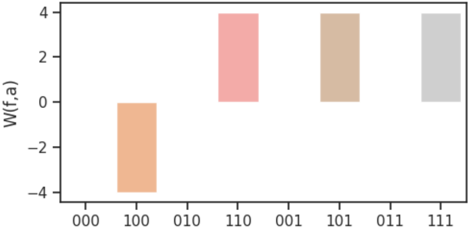
- Truth table: a 2^n-bit vector Ω_f specifying $f(x)$ for all $x \in \{0,1\}^n$

(x_1, x_2, x_3)	000	100	010	110	001	101	011	111
Ω_f	0	1	1	0	1	0	1	0

- Algebraic Normal Form (ANF): Sum (XOR) of products (AND)

$$f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3 \oplus x_2 x_3$$

- Walsh Transform: correlation with linear functions $a \cdot x$,
  $W(f,a) = \sum_{x \in \{0,1\}^n} (-1)^{f(x) \oplus a \cdot x}$ for all $a \in \{0,1\}^n$

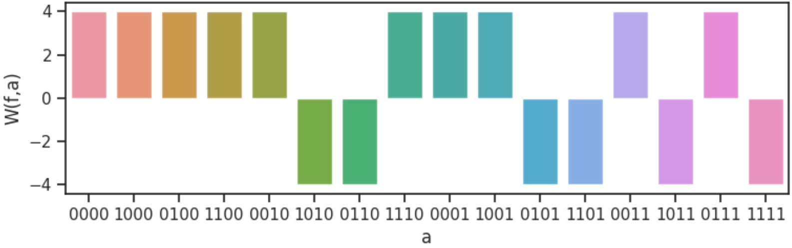


Bent Functions

- *Parseval's Relation*, valid on any Boolean function:

$$\sum_{a\in\{0,1\}^n} [W(f,a)]^2 = 2^{2n} \text{ for all } f:\{0,1\}^n \to \{0,1\}$$

- Bent functions: $W(f,a) = \pm 2^{\frac{n}{2}}$ for all $a \in \{0,1\}^n$
  - Reach the highest possible *nonlinearity*
  - Exist only for n even and they are *unbalanced*



Example: $f(x_1,x_2,x_3,x_4) = x_1x_3 + x_1x_4 + x_2x_4$

Constructions of Bent Functions

Given $n = 2m$:

- **Maiorana-McFarland** [M73]): $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is defined as

$$f(x, y) = x \cdot \pi(y) \oplus g(y)$$

where $\pi : \mathbb{F}_2^m \to \mathbb{F}_2^m$ is any permutation of $\mathbb{F}_2^m$ and $g : \mathbb{F}_2^m \to \mathbb{F}_2$ is any function of m variables

- **Partial spreads** [D74]: $f \in \mathcal{PS}^-$ ($f \in \mathcal{PS}^+$) is defined as

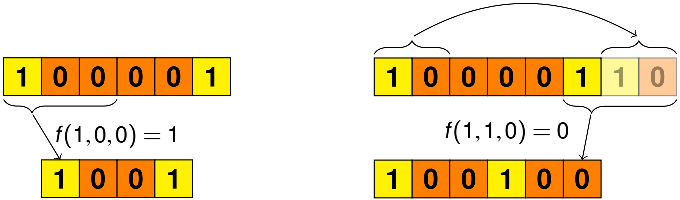
$$supp(f) = \bigcup_{S \in P} (S \setminus \{\underline{0}\}) \quad \left(supp(f) = \bigcup_{S \in P} S \right) ,$$

with $\mathcal{S}$ a family of 2^{m-1} $(+1)$ m-dimensional subspaces of $\mathbb{F}_2^n$ with pairwise trivial intersection

Part 1: Cellular Automata and Mutually Orthogonal Latin Squares

- A vectorial function $F : \mathbb{F}_q^n \to \mathbb{F}_q^m$ with *uniform* coordinates

Example: $q = 2$, $n = 6$, $d = 3$, $f(s_i, s_{i+1}, s_{i+2}) = s_i \oplus s_{i+1} \oplus s_{i+2}$



No Boundary CA – NBCA

Periodic Boundary CA – PBCA

- Each cell updates its state $s \in \{0,1\}$ by evaluating a local rule $f : \{0,1\}^d \to \{0,1\}$ on itself and the $d-1$ cells on its right

Mutually Orthogonal Latin Squares (MOLS)

Definition

A *Latin square* is a $n \times n$ matrix where all rows and columns are permutations of $[n] = \{1, \cdots, n\}$. Two Latin squares are *orthogonal* if their superposition yields all the pairs $(x, y) \in [n] \times [n]$.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

1 1	3 4	4 2	2 3
4 3	2 2	1 4	3 1
2 4	4 1	3 3	1 2
3 2	1 3	2 1	4 4

- k-**MOLS**: set of k pairwise orthogonal Latin squares
- k-MOLS are equivalent to $OA(n^2, k, n, 2)$

Latin Squares through Bipermutive CA (1/2)

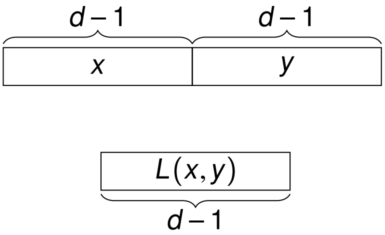
- Bipermutive CA: local rule f is defined as

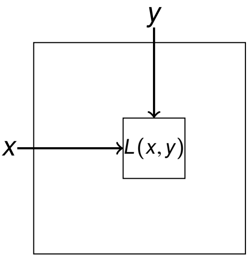
$$f(x_1,\cdots,x_d) = x_1 + \varphi(x_2,\cdots,x_{d-1}) + x_d$$

- $\varphi : \mathbb{F}_q^{d-2} \to \mathbb{F}_q$: generating function of f

Lemma ([E93, M16])

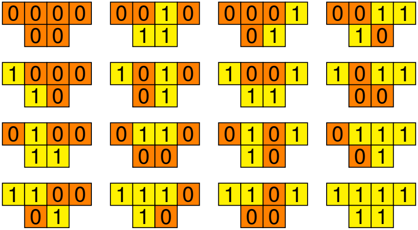
A CA $F : \mathbb{F}_q^{2(d-1)} \to \mathbb{F}_q^d$ with bipermutive rule $f : \mathbb{F}_q^d \to \mathbb{F}_q$ generates a Latin square of order $N = q^{d-1}$





Latin Squares through Bipermutive CA (2/2)

- Example: CA $F : \mathbb{F}_2^4 \to \mathbb{F}_2^2$, $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$ (Rule 150)
- Encoding: $00 \mapsto 1, 10 \mapsto 2, 01 \mapsto 3, 11 \mapsto 4$



(a) Rule 150 on 4 bits

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(b) Latin square L_{150}

Linear CA

- Local rule: *linear combination* of the neighborhood cells

$$f(x_1,\cdots,x_d) = a_1x_1 + \cdots + a_dx_d \ , \ a_i \in \mathbb{F}_q$$

- Associated polynomial:

$$f \mapsto p_f(X) = a_1 + a_2X + \cdots + a_dX^{d-1}$$

- Global rule: $F : \mathbb{F}_q^n \to \mathbb{F}_q^{n-d+1}$ is described by a $(n-d+1)\times n$ transition matrix:

$$M_F = \begin{pmatrix} a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_1 & \cdots & a_d \end{pmatrix}$$

$$x = (x_1,\cdots,x_n) \mapsto M_F x^\top$$

- **Remark:** a linear rule is bipermutive iff $a_1, a_d \neq 0$

Sylvester Matrices

- Two linear bipermutive CA with rules $f, g : \mathbb{F}_q^d \to \mathbb{F}_q$ generate orthogonal Latin squares iff the following matrix is invertible:

$$M_{F,G} = \begin{pmatrix} a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & \cdots & a_d & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_1 & \cdots & a_d \\ b_1 & \cdots & b_d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_1 & \cdots & b_d & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_1 & \cdots & b_d \end{pmatrix}$$

- ... but this is the **Sylvester matrix** of the two polynomials p_f, p_g, and $det(M_{F,G} \neq 0 \Leftrightarrow \gcd(p_f, p_g) = 1$

MOLS from Linear Bipermutive CA (LBCA)

Theorem ([MGLF20])

A set of t linear bipermutive CA $F_1, \ldots F_t : \mathbb{F}_q^{2(d-1)} \to \mathbb{F}_q^{d-1}$ generates a family of t-MOLS of order $N = q^{d-1}$ if and only if their associated polynomials are pairwise coprime

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

(a) Rule 150

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

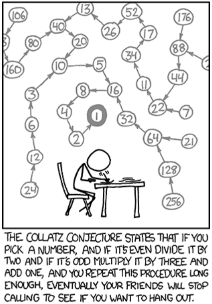
(b) Rule 90

1, 1	4, 2	3, 3	2, 4
2, 2	3, 1	4, 4	1, 3
4, 3	1, 4	2, 1	3, 2
3, 4	2, 3	1, 2	4, 1

(c) Superposition

Figure: $P_{150}(X) = 1 + X + X^2$, $P_{90}(X) = 1 + X^2$ (coprime)

Counting MOLS from linear CA



S `https://xkcd.com/710/`

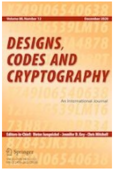
- Number of coprime polynomials over $\mathbb{F}_2$ of degree n and nonzero constant term:

$$a(n) = 4^{n-1} + a(n-1) = \frac{4^{n-1}-1}{3}$$
$$= 0, 1, 5, 21, 85, \ldots$$

- Corresponds to OEIS A002450

- Generalized to any finite field, along with size of largest family of pairwise coprime polynomials, in:

L. Mariot, M. Gadouleau, E. Formenti, and A. Leporati. Mutually orthogonal latin squares based on cellular automata. Des. Codes Cryptogr. 88(2):391–411 (2020)



Part 2: The Complicated Construction

Hadamard Matrices

- Hadamard Matrix: a $n \times n$ matrix with ± 1 entries and s.t. $H \cdot H^\top = I_n$

$$H = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}, \; n = 4$$

- Necessary condition: $n = 1, 2$ or $n = 4k$
- Hadamard Conjecture: a Hadamard matrix exists for every $n = 4k$

Theorem (Dillon, 1974 [D74])

Given $f : \{0,1\}^n \to \{0,1\}$ and $\hat{f}(x) = (-1)^{f(x)}$. Define the $2^n \times 2^n$ matrix H for all $x, y \in \{0,1\}^n$ as:

$$H(x,y) = \hat{f}(x \oplus y)$$

Then, f is a bent function if and only if H is a Hadamard matrix.

Example: $f(x_1, x_2) = x_1 x_2$

x_1	x_2	$x_1 x_2$
0	0	0
1	0	0
0	1	0
1	1	1

$$H = \begin{pmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{pmatrix}$$

Hadamard Matrices from MOLS

Orthogonal Array $OA(t,N)$ for t MOLS of order N: $N^2 \times t$ matrix where each column is the "linearization" of a Latin square

Theorem (Bush, 1973 [B73])

Given a set of t MOLS of order $N = 2t$, and A the associated $OA(t,2t)$, define the $4t^2 \times 4t^2$ matrix H as follows:

$$H(i,j) = \begin{cases} +1 \ , & \text{if } i = j \\ -1 \ , & \text{if } i \neq j \text{ and } \exists k \in \{1,\cdots,t\} \text{ s.t. the column} \\ & k \text{ of } A \text{ has the same symbol in rows } i \text{ and } j \\ +1 \ , & \text{otherwise} \end{cases}$$

for $i,j \in \{1,\cdots,4t^2\}$. Then, H is a symmetric Hadamard matrix.

Bent Functions from any MOLS?

- **Remark**: Not all t-MOLS sets give rise to a Hadamard matrix with the $\hat{f}(x \oplus y)$ structure required for a bent function!
- Smallest counterexample: $n = 6$, $t = 2^{\frac{n-2}{2}} = 4$, $N = 2t = 8$

1	2	3	4	5	8	6	7
2	1	4	3	8	5	7	6
3	4	1	2	7	6	8	5
4	3	2	1	6	7	5	8
5	8	7	6	1	2	4	3
8	5	6	7	2	1	3	4
6	7	8	5	4	3	1	2
7	6	5	8	3	4	2	1

(a) L_1

1	2	3	4	5	6	7	8
3	4	1	2	8	7	6	5
5	6	8	7	1	2	4	3
8	7	5	6	3	4	2	1
4	3	2	1	7	8	5	6
2	1	4	3	6	5	8	7
6	5	7	8	2	1	3	4
7	8	5	6	4	3	1	2

(b) L_2

1	2	3	4	5	6	7	8
4	3	2	1	7	8	5	6
8	7	5	6	3	4	2	1
6	5	7	8	2	1	3	4
7	8	6	5	4	3	1	2
5	6	8	7	1	2	4	3
3	4	1	2	8	7	6	5
2	1	4	3	6	5	8	7

(c) L_3

1	2	3	4	5	6	7	8
5	6	8	7	1	2	4	3
4	3	2	1	7	8	5	6
7	8	6	5	4	3	1	2
8	7	5	6	3	4	2	1
3	4	1	2	8	7	6	5
2	1	4	3	6	5	8	7
6	5	7	8	2	1	3	4

(d) L_4

- The resulting 64×64 Hadamard matrix does not give a bent function

From Linear CA to Bent Functions

- Question: Are the MOLS arising from linear CA suitable for constructing bent functions?
- We consider only CA over $\mathbb{F}_q$ with $q = 2^l$, $l \in \mathbb{N}$
- The order of the Hadamard matrix must be $4t^2 = 2^n$
- We need t coprime polynomials of degree $b = d - 1$:

$$2^{lb} = 2t \Leftrightarrow lb = 1 + \log_2 t$$

- Since both l and b are integers, $t = 2^w$ for $w \in \mathbb{N}$

From Linear CA to Bent Functions

Theorem

Let H be the Hadamard matrix of order $2^{2(w+1)}$ defined by the t LBCA $F_1, \cdots F_t : \mathbb{F}_q^{2b} \to \mathbb{F}_q^b$, and define $f : \mathbb{F}_2^n \to \mathbb{F}_2$, $n = 2(w+1)$ as:

$$f(x) = \begin{cases} 0 \ , & \text{if } x = 0 \\ 1 \ , & \text{if } x \neq 0 \text{ and } \exists k \in \{1, \cdots, t\} \text{ s.t. } F_k(x) = 0 \\ 0 \ , & \text{otherwise} \end{cases} ,$$

Then, it holds that:

$$H(x,y) = \hat{f}(x \oplus y)$$

and thus f is a bent function

Remark: The linearity of the CA is crucial to grant this result (and costed us our first reject!)

Example

$p_f(X) = 1 + X^2$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$p_g(X) = 1 + X + X^2$

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

$$A = \begin{array}{cc} L_1 & L_2 \\ \hline 1 & 1 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 2 & 2 \\ 1 & 3 \\ 4 & 4 \\ 3 & 1 \\ 3 & 4 \\ 4 & 1 \\ 1 & 2 \\ 2 & 3 \\ 4 & 3 \\ 3 & 2 \\ 2 & 1 \\ 1 & 4 \end{array}$$

$$H = \begin{pmatrix}
+ & + & + & + & + & - & - & + & + & + & - & - & + & - & + & - \\
+ & + & + & + & - & + & + & - & + & + & - & - & - & + & - & + \\
+ & + & + & + & - & + & + & - & - & - & + & + & + & - & + & - \\
+ & + & + & + & + & - & - & + & - & - & + & + & - & + & - & + \\
+ & - & - & + & + & + & + & + & + & - & + & - & + & + & - & - \\
- & + & + & - & + & + & + & + & - & + & - & + & + & + & - & - \\
- & + & + & - & + & + & + & + & + & - & + & - & - & - & + & + \\
+ & - & - & + & + & + & + & + & - & + & - & + & - & - & + & + \\
+ & + & - & - & + & - & + & - & + & + & + & + & + & - & - & + \\
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- & - & + & + & - & + & - & + & + & + & + & + & + & - & - & + \\
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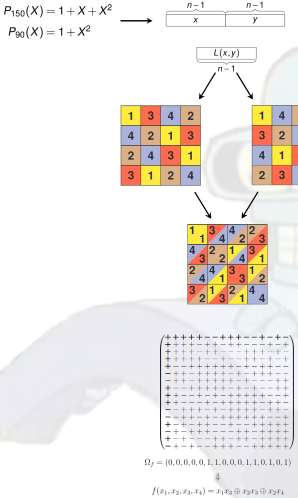
+ & - & + & - & - & - & + & + & - & + & + & - & + & + & + & + \\
- & + & - & + & - & - & + & + & + & - & - & + & + & + & + & +
\end{pmatrix}$$

$$\Omega_f = (0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1)$$

$$\Downarrow$$

$$f(x_1, x_2, x_3, x_4) = x_1 x_3 \oplus x_2 x_3 \oplus x_2 x_4$$

Figure 3: Example of bent function of $n = 4$ variables generated by the $t = 2$ MOLS of order $2t = 4$ defined by the LBCA with rule 90 and 150, respectively. The two Latin squares are represented on the left in the OA form. The first row and the first column of the Hadamard matrix H coincide with the polarity truth table of the function.



Combinatorial questions addressed in [GMP20]:

- **Existence:** for even *n*, does a large enough family of coprime polynomials exist?
- **Counting:** how many families of this kind exist (= number of CA-based bent functions)?

Existence and Counting

Theorem ([GMP20])

Let $l, b, w \in \mathbb{N}$ such that $lb = 1 + w$, and $q = 2^l$. Then:

- *There is a family of $t = 2^w$ pairwise coprime polynomials of degree b over $\mathbb{F}_q$ if and only if $b \in \{1, 2\}$*
- *The number of bent functions of $n = 2(w+1)$ variables that can be obtained by Theorem 4 is:*
  - $\binom{2^{w+1}-1}{2^w}$, *when $b = 1$*
  - $\sum_{A=0}^{l_2} \binom{l_2}{A} \sum_{B=0}^{2^w - A} \binom{l_1}{B} \binom{l_1 - B}{2(2^w - B - A)} \frac{(2(2^w - B - A))!}{(2^w - B - A)! 2^{2^w - B - A}}$, *where $l_2 = \frac{1}{2}(q^2 - q)$ and $l_1 = q - 1$, when $b = 2$.*

Remark: each family can always be augmented with the polynomials 1 and X^b

Part 3: A Simplified Construction with Linear Recurring Sequences

Reviewers Feedback

- First attempt: BFA, reject (incomplete proof)
- Second attempt: CCDS, reject (complicated construction, no guarantee the obtained bent functions are new)
- Third attempt: DESI, major revision

Strictest (and most enthusiastic!) review:

This paper must be published in some form! :)
It has the potential of becoming a major reference on bent functions because it identifies a new source of partial spreads large enough to give bent functions!

...

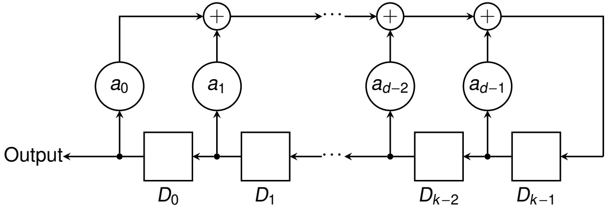
But not in its present form which buries and obscures their great new result in many pages of material on subjects which, in retrospect, are unnecessary for a lucid exposition of their new results. It may be in order to *briefly* mention how they were led to their theorem via the CA, Latin square, MOLS and Bush

Linear Recurring Sequences (LRS)

- Sequence $\{x_i\}_{i\in\mathbb{N}}$ satisfying the following relation:

$$a_0 x_i + a_1 x_{i+1} + \ldots + a_{d-1} x_{i+d-1} = x_{i+d}$$

- Computed by a *Linear Feedback Shift Register* (LFSR):



- Feedback polynomial:

$$f(X) = a_0 + a_1 X + \cdots a_{d-1} X^{d-1} + X^d$$

Linear map associated to a LRS

- Let $\mathcal{S}(f(X))$ be the set of all sequences satisfying a linear recurrence with feedback polynomial $f(X)$
- Take the *projection* these sequences onto their $2d$ coordinates
- We obtain a d-dim subspace $S_f \subseteq \mathbb{F}_q^{2d}$ which is the kernel of the linear map $F : \mathbb{F}_q^{2d} \to \mathbb{F}_q^{d}$:

$$F(x_0, \cdots, x_{2d-1})_i = a_0 x_i + a_1 x_{i+1} + \ldots + a_{d-1} x_{i+d-1} + x_{i+d} \ ,$$

with matrix

$$M_F = \begin{pmatrix} a_0 & \cdots & a_{d-1} & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_0 & \cdots & a_{d-1} & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & a_0 & \cdots & a_{d-1} & 1 \end{pmatrix}$$

- ... but this is *exactly* the global rule of a linear CA!

Partial Spreads from Coprime Polynomials

Lemma

Given $f, g \in \mathbb{F}_q[X]$ over $\mathbb{F}_q$ of degree $d \geq 1$, defined as:

$$f(X) = a_0 + a_1 X + \cdots + a_{d-1} X^{d-1} + X^d \ , \tag{1}$$

$$g(X) = b_0 + b_1 X + \cdots + b_{d-1} X^{d-1} + X^d \ , \tag{2}$$

Then, the kernels of $F, G : \mathbb{F}_q^{2d} \to \mathbb{F}_q^d$ have trivial intersection if and only if $\gcd(f, g) = 1$

Consequence: a family of t pairwise coprime polynomials defines a partial spread

Equivalence to $\mathcal{PS}_{ap}$ functions for degree $b = 1$

- *Bivariate form* of the Desarguesian spread for $n = 2m$ [M16]:

$$DS = \{E_a \subseteq \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} : \ a \in \mathbb{F}_{2^m}\} \cup E_\infty \quad \text{, where :}$$
$$E_a = \{(x, ax) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} : \ x \in \mathbb{F}_{2^m}\} \ ,$$
$$E_\infty = \{(0, y) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} : \ y \in \mathbb{F}_{2^m}\} \ .$$

- Form of the linear map F_i when $b = 1$: $F_i(x_0, x_1) = a_i x_0 + x_1$
- Kernel of F_i: $ker(F_i) = \{(x_0, x_1) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} : x_1 = a_i x_0\} = \{(x, a_i x) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} : x \in \mathbb{F}_{2^m}\} = E_{a_i} \in DS$

Lemma

Our construction coincides with the class $\mathcal{PS}_{ap}$ when $b = 1$.

2-Rank Distributions for $n = 8$, $b = 1$

- needed: $t = 8$ pairwise coprime polynomials of deg. 1 on $\mathbb{F}_{2^4}$
- No need to check for $\mathcal{PS}^+$ in this case
- # $\mathcal{PS}^-$ functions: $\binom{17}{8} = 24310$
- **2-rank**: rank of the *translate design* $f(x \oplus y)$

Rank	#Functions
30	510
36	4080
40	2040
42	17680
Total	24310

- **Main result**: verification (and extension) of Weng et al.'s counting results [W07]

2-Rank Distributions for $n = 8$, $b = 2$

Type	Rank	#Functions
$\mathcal{PS}^-$	36	20
	40	24
	42	28
	44	123
	46	78
	Total	273
$\mathcal{PS}^+$	40	45
	44	19
	46	18
	Total	82

- Ingredients: $t = 8$ pairwise coprime polynomials of degree $b = 2$ over $\mathbb{F}_{2^2}$ ($t + 1 = 9$ for $\mathcal{PS}^+$ functions)
- \# $\mathcal{PS}^-$ functions: 273
- \# $\mathcal{PS}^+$ functions: 82
- **1st Finding**: *none* of these functions is in $\mathcal{M}^\#$ ($UB_\mathcal{M} = 30$)
- **2nd Finding**: many of these functions (rank > 42) are not even in $\mathcal{PS}_{ap}$

Conclusions

Recap and Open Problems

Remarkable findings:

- (very convoluted) construction of bent functions via CA, Latin Squares and Hadamard matrices
- Simplification based on kernels of LRS subspaces
- Resulting bent functions coincide with $\mathcal{PS}_{ap}$ for degree $b = 1$
- For $b = 2$, many functions are not in $\mathcal{PS}_{ap}$

Open problems:

- Are functions stemming from polynomials of degree $b = 2$ really new?
- Implementation of CA-based bent functions via LFSR [ML18]
- Is it possible to get more functions without constraints on the constant term of the polynomials?

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